## Mechanical quantities



- Temperature (NVE MD):

$$T_{\text{kin}} = \frac{E_{\text{kin}}}{fk_B/2}$$

notation here:
$U = U(r^N)$ = potential energy
$E = E(T, V)$ = internal energy
$f$ = # of degrees of freedom

- Internal energy:

$$E = \langle E_{\text{kin}} + U \rangle \overset{NVT}{=} \frac{f}{2}k_BT + \langle U \rangle \equiv E_{\text{id}} + E_{\text{res}}$$

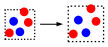
res = residual
see next slide

- Pressure

$\beta = 1/k_BT$

$$p = \frac{N}{V}k_BT - \left\langle \left( \frac{\partial U(V^{1/3}\vec{\xi}^N)}{\partial V} \right)_{\vec{\xi}^N} \right\rangle \equiv p_{\text{id}} + p_{\text{res}}$$

– dimensionless (scaled) coordinates $\vec{\xi}_i$: $\vec{r}_i = V^{1/3}\vec{\xi}_i$
– red derivative is calculated at constant $\vec{\xi}^N$,
whole configurations is uniformly shrank/swelled
– $p_{\text{id}}$ = kinetic contribution (=ideal gas), also $p_{\text{id}} = \phi 2E_{\text{kin}}/3V$,
where in periodic b.c. $\phi = N/(N-1)$ takes into acccount 3 zero conserved momenta
– $p_{\text{res}}$ = cohesion contribution



## Pressure tensor



Also called stress tensor (in solids):

$$\overleftrightarrow{P} = \overleftrightarrow{P}_{\text{id}} + \overleftrightarrow{P}_{\text{res}} = \frac{1}{V}\sum_{i=1}^{N}\left(\phi m_i \vec{v}_i \vec{v}_i + \vec{f}_i \vec{r}_i\right)$$

- Tensor product $\overleftrightarrow{T} = \vec{u}\vec{v}$, also denoted $\overleftrightarrow{T} = \vec{u} \otimes \vec{v}$: $T_{ab} = v_a v_b$.
- $\phi = N/(N-1)$ takes into acccount 3 zero conserved momenta in periodic b.c.
- The scalar pressure is 1/3 of its trace, $p = \text{tr}(\overleftrightarrow{P})/3$.
- In models with rigid (constrained) bonds, the virial of constrained forces has to be included (it depends on velocities).
- Diagonal terms are good for calculating surface tension.
- Off-diagonal terms are good for calculating viscosity.*
- Can be also calculated by the virtual box change:
  – elongation in $\hat{x} \Rightarrow P_{xx}$,
  – simulation cell shape change ⇒ off-diagonal terms.

*Suprisingly, the diagonal terms can be used to calculate viscosity, too.

## Pressure in the *NVT* ensemble: Derivation

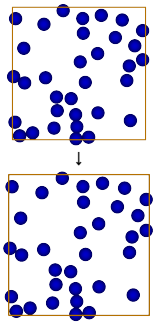


$\text{d}F = -S\text{d}T - p\text{d}V \qquad \langle X \rangle = \frac{1}{Q_N}\int_{V^N} X(\vec{r}^N)\exp[-\beta U(\vec{r}^N)]\text{d}\vec{r}^N$

$$p = -\left(\frac{\partial F}{\partial V}\right)_T, \quad F = -k_BT \ln \frac{Q_N}{N!\Lambda^{3N}}$$

$$Q_N = \int_{V^N} \exp[-\beta U(\vec{r}^N)]\text{d}\vec{r}^N \overset{\vec{r}_i = V^{1/3}\vec{\xi}_i}{=} \int_{1^{3N}} \exp[-\beta U(V^{1/3}\vec{\xi}^N)]V^N \text{d}\vec{\xi}^N$$

$$\begin{aligned} p &= -\left(\frac{\partial F}{\partial V}\right)_T = k_BT\left(\frac{\partial \ln Q_N}{\partial V}\right)_{\vec{\xi}^N} = \frac{k_BT}{Q_N}\left(\frac{\partial Q_N}{\partial V}\right)_{\vec{\xi}^N} \\ &= \frac{k_BT}{Q_N}\int_{1^{3N}} \exp[-\beta U(V^{1/3}\vec{\xi}^N)]NV^{N-1}\text{d}\vec{\xi}^N \\ &\quad + \frac{k_BT}{Q_N}\int_{1^{3N}} \exp[-\beta U(V^{1/3}\vec{\xi}^N)](-\beta)\left(\frac{\partial U(V^{1/3}\vec{\xi}^N)}{\partial V}\right)_{\vec{\xi}^N} V^N \text{d}\vec{\xi}^N \\ &= \frac{N}{V}k_BT - \left\langle \left(\frac{\partial U(V^{1/3}\vec{\xi}^N)}{\partial V}\right)_{\vec{\xi}^N} \right\rangle = \boxed{\text{ideal part (kinetic)}} + \boxed{\text{residual part (correction)}} \end{aligned}$$



## Surface tension of liquid

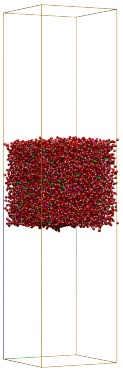


Using slab geometry – elongated periodic box (2:1 or more)

$$\gamma = \left(\frac{\partial F}{\partial \mathcal{A}}\right)_{V,T} = \left(\frac{\partial G}{\partial \mathcal{A}}\right)_{p,T} = -\frac{1}{4}L_z P_{\text{t}}, \quad \text{where } P_{\text{t}} = P_{xx} + P_{yy} - 2P_{zz}.$$

- It is "mechanical quantity".
- Cannot be used for interfacial energy of crystals (it is "entropic quantity").
- Cutoff corrections – several variants (cf. simen09).
- $P_{zz}$ = saturated vapor pressure;
if small enough, $P_{xx}, P_{yy}$ can be replaced by the usual pressure $p$.
- Alternative: virtual area change method [Gloor et al.: JCP 123, 134703 (2005)]:
scaling in $\hat{x}, \hat{y}$ and inversely squared in $\hat{z}$ so that volume is preserved:

$$\gamma = \left\langle \left(\frac{\delta U}{\delta \mathcal{A}}\right)_V \right\rangle$$



## Residual quantities



= with respect to the standard state of ideal gas at the same temperature, volume (= density), and composition as the given system. Usefull in the canonical (NVT) ensemble.

sometimes "excess"; for solvation Gibbs energy or $\mu$ of solute, also "Ben-Naim standard state"

For the Helmholtz energy:

ideal gas: $Q_N = V^N$

$$F = -k_BT \ln Z_N = -k_BT \ln \frac{Q_N}{N!\Lambda^{3N}} = -k_BT \ln \frac{V^N}{N!\Lambda^{3N}} - k_BT \ln \frac{Q_N}{V^N} \equiv F_{\text{id}} + F_{\text{res}}$$

**Refresh:**
Thermal de Broglie wavelength:

$$\Lambda = \frac{h}{\sqrt{2\pi m k_BT}}$$

chemical potential of ideal gas:

$$\mu_{\text{id}} = \left(\frac{\partial F_{\text{id}}}{\partial N}\right)_{T,V} = k_BT \ln \frac{N\Lambda^3}{V}$$

## Fluctuation quantities



$\langle(\Delta X)^2\rangle = \text{Var}\,X$ = (mean quadratic) fluctuation = variance $\qquad \Delta X = X - \langle X \rangle$

fluctuation = (mech/el/. . . quantity)′ = (thermodynamic potential)″

**Less accurate than mean values!**

E.g. (NVT): $P = -\left(\frac{\partial F}{\partial V}\right)_T$, $E = \left(\frac{\partial \beta F}{\partial \beta}\right)_V$, then:

- Heat capacity at constant $[V]$:

$$C_V = \left(\frac{\partial E}{\partial T}\right)_V = \frac{1}{k_BT^2}\text{Var}\,E = \frac{1}{k_BT^2}\langle(\Delta E_{\text{kin}} + \Delta U)^2\rangle$$

$\text{Cov}(U, E_{\text{kin}}) = \langle \Delta U \Delta E_{\text{kin}} \rangle = 0$, $\text{Var}\,E_{\text{kin}} = \frac{f}{2}(k_BT)^2$ (see exercise) ⇒

$$C_V = \frac{1}{k_BT^2}\left\langle \left(\frac{fk_BT}{2} + \Delta U\right)^2 \right\rangle = \frac{fk_B}{2} + \frac{1}{k_BT^2}\langle(\Delta U)^2\rangle \equiv C_{V,\text{id}} + C_{V,\text{res}}$$

- Isothermal compressibility – in the NPT ensemble

$$\kappa = -\frac{1}{V}\left(\frac{\partial V}{\partial P}\right)_T = \frac{\langle(\Delta V)^2\rangle}{Vk_BT}$$

Permittivity can be calculated from the fluctuation of the cell dipole moment, $\langle M^2 \rangle$, (more later).

## Pressure – virtual volume change method



$$p = \frac{N}{V}k_BT - \left\langle \left(\frac{\partial U(V^{1/3}\vec{\xi}^N)}{\partial V}\right)_{\vec{\xi}^N} \right\rangle$$

**Numerical derivative** (for a selected series of configurations)

$$\frac{\partial U}{\partial V} = \frac{U(V+\Delta V) - U(V)}{\Delta V} + \mathcal{O}(\Delta V) \equiv \frac{U(\text{[image]}) - U(\text{[image]})}{\Delta V} + \mathcal{O}(\Delta V)$$

$$\frac{\partial U}{\partial V} = \frac{U(V+\Delta V) - U(V-\Delta V)}{2\Delta V} + \mathcal{O}(\Delta V^2)$$

**Implementation:** $U(V + \Delta V)$ means that the whole configuration (all distances) is swelled by the same ratio; for molecules w.r.t. a reference point (then, $N$ = # of molecules):

$$\left(\frac{V + \Delta V}{V}\right)^{1/3}.$$

The scaled configuration is not included in the trajectory.

For models with a hard core such that swelling the box cannot cause an overlap, shrinking can be used: $P = Nk_BT/V + \frac{k_BT}{\Delta V}\langle e^{-[U(V-\Delta V)-U(V)]/k_BT}\rangle + \mathcal{O}(\Delta V)$

## Exercise



Calculate $\langle E_{\text{kin}} \rangle$ and $\text{Var}(E_{\text{kin}})$ for one degree of freedom, $E_{\text{kin}} = \frac{1}{2}mv^2$.

$$\begin{aligned} E_{\text{kin}} &= \frac{1}{2}mv^2 \\ \boldsymbol{\pi}(v) &= \frac{\exp(-\frac{1}{2}mv^2/k_BT)}{\int \exp(-\frac{1}{2}mv^2/k_BT)\text{d}v} \\ \langle E_{\text{kin}} \rangle &= \int E_{\text{kin}}\boldsymbol{\pi}(v)\text{d}v \\ \text{Var}(E_{\text{kin}}) &= \int (E_{\text{kin}} - \langle E_{\text{kin}} \rangle)^2 \boldsymbol{\pi}(v)\text{d}v \\ &= \frac{1}{2}(k_BT)^2 \end{aligned}$$

```
> restart;
> assume(m>0,k>0,T>0);
> K:=m/2*v^2: kT:=k*T:
> inorm:=int(exp(-K/kT),v=-infinity..infinity);
                    inorm := sqrt(2) k~ T~ sqrt(pi) / sqrt(m~ k~ T~)
> averK:=int(K*exp(-K/kT),v=-infinity..infinity)/inorm;
                    averK := 1/2 k~ T~
> fluctK:=int((K-averK)^2*exp(-K/kT),v=-infinity..infinity)/inorm;
                    fluctK := 1/2 k~^2 T~^2
```

## Pressure from the virial of force



**The derivative expanded:**

$$\frac{\partial U(V^{1/3}\vec{\xi}^N)}{\partial V} = \sum_{i=1}^{N}\frac{1}{3}V^{-2/3}\vec{\xi}_i \cdot \frac{\partial U}{\partial \vec{r}_i} = \frac{1}{3V}\sum_{i=1}^{N}\vec{r}_i \cdot \frac{\partial U}{\partial \vec{r}_i}$$

($-\vec{f}_i$)

The result is

$$p = \frac{N}{V}k_BT + \frac{1}{3V}\langle W_f \rangle, \qquad W_f = -\sum_{i=1}^{N}\vec{r}_i \cdot \frac{\partial U}{\partial \vec{r}_i} = \sum_{i=1}^{N}\vec{r}_i \cdot \vec{f}_i \quad \text{(virial of force)}$$

. . . cannot be directly applied in the periodic boundary conditions.

- Pair additivity in the periodic boundary conditions ⇒

$$p = \frac{N}{V}k_BT - \frac{1}{3V}\sum_{i<j}\langle r_{ij}u'(r_{ij})\rangle \equiv p_{\text{id}} + p_{\text{res}}$$

- For molecular models one can use either the atom-frame ($N$ = # of atoms/sites), or molecular frame ($N$ = # of molecules; typically, ref. point = center of mass). Of course, the formulas differ.

## Isochoric heat capacity $C_V$ in the NVE ensemble



In the **canonical** ensemble:

$$\text{Var}\,X = k_BT^2\frac{\partial X}{\partial T} \equiv k_BT^2 X' \quad (X = U, E_{\text{kin}}) \qquad (1)$$

$$\text{Cov}(U, E_{\text{kin}}) = 0$$

$$C_V = U' + E'_{\text{kin}}, \quad E'_{\text{kin}} = \frac{fk_B}{2}$$

Unnormalized probability distribution for temperature $T$:

$$w(U, E_{\text{kin}}) \sim \exp\left[-\frac{(\delta U)^2}{2\,\text{Var}\,U} - \frac{(\delta E_{\text{kin}})^2}{2\,\text{Var}\,E_{\text{kin}}}\right]$$

where the deviations are linearized: $(U, E_{\text{kin}}) = (U_0 + \delta U, E_{\text{kin},0} + \delta E_{\text{kin}})$

Unnormalized probability distribution taking into account change in temperature, $T = T_0 + \delta T$:

$$w(T, U, E_{\text{kin}}) \sim \exp\left[-\frac{(\delta U - U'\delta T)^2}{2\,\text{Var}\,U} - \frac{(\delta E_{\text{kin}} - E'_{\text{kin}}\delta T)^2}{2\,\text{Var}\,E_{\text{kin}}}\right]$$

## Isochoric heat capacity $C_V$ in the NVE ensemble II



In the **MD NVE** ensemble it holds $U + E_{\text{kin}} = E = \text{const}$ or $U = E - E_{\text{kin}}$, and of course $\delta E_{\text{kin}} = -\delta U$, so that let us choose $\delta E_{\text{kin}}$ as the independent variable

$$w(T, E_{\text{kin}}) \sim \exp\left[-\left(\frac{1}{2k_{\text{B}}T^2U'} + \frac{1}{2k_{\text{B}}T^2E'_{\text{kin}}}\right)\delta E_{\text{kin}}^2 - \left(\frac{U'}{2k_{\text{B}}T^2} + \frac{E'_{\text{kin}}}{2k_{\text{B}}T^2}\right)\delta T^2\right]$$

where the variances were replaced by derivatives, see (1).

NB: term $\delta E_{\text{kin}}\delta T$ cancels out!

From term at $\delta T^2$:

$$\text{Var}\, T = \frac{k_{\text{B}}T^2}{U' + E'_{\text{kin}}} = \frac{k_{\text{B}}T^2}{C_V}$$

which is a known expression for the temperature in the NVE ensemble, however, $T$ here is **not** the kinetic temperature, but $T$ in formula $\mathrm{d}E = T\mathrm{d}S\,[V]$. The average temperature is the same, but not the fluctuation.

## Examples



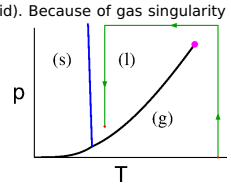
- For $\lambda = \beta$ we get as before:

$$\beta_1 F_{\text{res}}(\beta_1) - \beta_0 F_{\text{res}}(\beta_0) = \int_{\beta_0}^{\beta_1} \langle U\rangle \mathrm{d}\beta$$

- Integration from an Einstein crystal to a real crystal.
  NB: Einstein crystal = independent (here classical) harmonic oscillators at lattice sites.
  There are minor solvable problems when the crystal is detached from the fixed sites.
- Integration from ideal gas (e.g., around the critical point to liquid). Because of gas singularity at zero density, one of the integrals is (for $NPT$):

$$\ln\phi = \frac{\mu - \mu^\circ}{RT} = \int_0^p \left(V_{\text{m}} - \frac{RT}{p'}\right)\mathrm{d}p'$$

  It helps to use a virial equation of state at low densities.




## Isochoric heat capacity $C_V$ in the NVE ensemble III



From the term at $\delta E^2$:

$$\text{Var}\, E = \text{Var}\, U = \text{Var}\, E_{\text{kin}} = \frac{k_{\text{B}}T^2}{1/U' + 1/E'_{\text{kin}}} = \frac{k_{\text{B}}T^2}{1/(C_V - E'_{\text{kin}}) + 1/E'_{\text{kin}}}$$

Since $E'_{\text{kin}}$ is known:

$$C_V = \frac{fk}{2}\left[\left(\frac{2\,T^2}{f\,\text{Var}\, T_{\text{kin}}} - 1\right)^{-1} + 1\right]$$

where $T_{\text{kin}} \equiv E_{\text{kin}}/(fk/2)$ is the kinetic temperature.

Final expression applicable in simulations:

$$C_V = \frac{fk}{2}\left[\left(\frac{2\,\langle T_{\text{kin}}\rangle^2}{f\,\langle(\Delta T_{\text{kin}})^2\rangle} - 1\right)^{-1} + 1\right]$$

## Non-Boltzmann sampling



We want $(\beta U)_1$, but we simulate $(\beta U)_0$ (can change $\beta$/U/both)[†]

$$\Delta(\beta U) = (\beta U)_1 - (\beta U)_0$$

$$\langle X\rangle_{(\beta U)_1} = \frac{\int X \mathrm{e}^{-(\beta U)_1}\mathrm{d}\vec{r}^N}{\int \mathrm{e}^{-(\beta U)_1}\mathrm{d}\vec{r}^N} = \frac{\frac{1}{Q_0}\int X \mathrm{e}^{-(\beta U)_0}\mathrm{e}^{-\Delta(\beta U)}\mathrm{d}\vec{r}^N}{\frac{1}{Q_0}\int \mathrm{e}^{-(\beta U)_0}\mathrm{e}^{-\Delta(\beta U)}\mathrm{d}\vec{r}^N} = \frac{\langle X\mathrm{e}^{-\Delta(\beta U)}\rangle_0}{\langle \mathrm{e}^{-\Delta(\beta U)}\rangle_0}$$

Helmholtz energy:

$$\begin{aligned}
\Delta(\beta F_{\text{res}}) &= \beta_1 F_{\text{res}}((\beta U)_1) - \beta_0 F_{\text{res}}((\beta U)_0) \\
&= -\ln\left(\frac{Q_1}{Q_0}\right) = -\ln\frac{\int \mathrm{e}^{-(\beta U)_1}\mathrm{d}\vec{r}^N}{\int \mathrm{e}^{-(\beta U)_0}\mathrm{d}\vec{r}^N} \\
&= -\ln\frac{\int \mathrm{e}^{-(\beta U)_0}\mathrm{e}^{-\Delta(\beta U)}\mathrm{d}\vec{r}^N}{\int \mathrm{e}^{-(\beta U)_0}\mathrm{d}\vec{r}^N} = -\ln\langle \mathrm{e}^{-\Delta(\beta U)}\rangle_0 \\
&= \ln\langle \mathrm{e}^{+\Delta(\beta U)}\rangle_1
\end{aligned}$$

where the last equation follows from $0 \longleftrightarrow 1$ interchange

[†] First usage probably Zwanzig (1954) "thermodynamic perturbation method", loosely also "umbrella sampling"

## Entropic quantities



These include $F$, $G$ ((⇐ partition function), $S$ (number of states $W$), $\mu$, $\Delta G$,...
They cannot be expressed as a simple $\langle\cdot\rangle$.

**Usage:** equilibria in general; solubility, ligand–receptor binding, stability of biomolecules, ...

- thermodynamic integration:
  - over a real variable ($T$, $V$, $P$)
  - over a coupling parameter
- reversible work calculated by the integration of force
- Widom particle insertion method;
  gradual insertion, alchemical transmutation
- non-Boltzmann sampling:
  - umbrella sampling
  - multiple histogram reweighting
  - metadynamics/Wang–Landau/conformational flooding
- local density method

Formally (for the configurational integral):

$$Q = \frac{\int \mathrm{e}^{-\beta U}\mathrm{d}\vec{r}^N}{1}$$

$$= \frac{\int \mathrm{e}^{-\beta U}\mathrm{d}\vec{r}^N}{V^{-N}\int \mathrm{e}^{-\beta U}\mathrm{e}^{+\beta U}\mathrm{d}\vec{r}^N} = \frac{V^N}{\langle \mathrm{e}^{\beta U}\rangle}$$

...totally useless

## Non-Boltzmann sampling contd.



- $\Delta(\beta U)$ must not be too large
- the thermodynamic integration is recovered for infinitesimally small $\Delta(\beta U)$:

$\mathrm{e}^x = 1 + x + \cdots$
$\ln(1+x) = x + \cdots$

$$\begin{aligned}
\Delta(\beta F_{\text{res}}) &= -\ln\langle \mathrm{e}^{-\Delta(\beta U)}\rangle_0 \\
&\approx -\ln\langle 1 - \Delta(\beta U)\rangle_0 \\
&= -\ln(1 - \langle\Delta(\beta U)\rangle_0) \\
&\approx \langle\Delta(\beta U)\rangle
\end{aligned}$$

$$\text{general } \lambda: \quad \partial(\beta F_{\text{res}})/\partial\lambda = \langle\partial\Delta(\beta U)/\partial\lambda\rangle_\lambda$$

$$\lambda = \beta: \quad \partial(\beta F_{\text{res}})/\partial\beta = \langle U\rangle$$


https://positiveoutlookblog.com/2015/01/30/surviving-the-storm-together

**Umbrella sampling**

The system in the middle is sampled: mid = $(\beta U)_0 + \Delta(\beta U)/2 = (\beta_0 U_0 + \beta_1 U_1)/2$:

$$\Delta(\beta F_{\text{res}}) = \ln\langle \mathrm{e}^{+\Delta(\beta U)/2}\rangle_{\text{mid}} - \ln\langle \mathrm{e}^{-\Delta(\beta U)/2}\rangle_{\text{mid}}$$

Nowadays, term "umbrella sampling" usually refers to many steps of similar kind.

**Scaled particle**

A particle growing step-wise, similar is "alchemic transmutation"

## Thermodynamic integration



Physical chemistry: $\mathrm{d}F = -S\mathrm{d}T - p\mathrm{d}V$, $\mathrm{d}G = -S\mathrm{d}T + V\mathrm{d}p$

$U = U(r^N)$ = potential energy
$E = E(T, V)$ = internal energy

Canonical ensemble ($\beta = 1/k_{\text{B}}T$):

$$\left(\frac{\partial F}{\partial V}\right)_T = -p \;\Rightarrow\; F_1 - F_0 = -\int_{V_0}^{V_1} p\mathrm{d}V, \quad \left(\frac{\partial(\beta F)}{\partial\beta}\right)_V = E \text{ or } \left(\frac{\partial(\beta F_{\text{res}})}{\partial\beta}\right)_V = \langle U\rangle$$

- Numerically integrated: $p$, $E$ must be determined at many points
- Start from a suitable reference state (known state, ideal gas, harmonic crystal)

Proof # 1 of $\frac{\partial(\beta F)}{\partial\beta} = E$:

$$\frac{\partial(\beta F)}{\partial\beta} = \frac{\partial(F/T)}{\partial(1/T)} = \frac{\partial(F/T)}{\partial T}\Big/\frac{\partial(1/T)}{\partial T} = \frac{-ST - F}{T^2}\Big/\left(\frac{-1}{T^2}\right) = ST + F = E$$

Proof # 2 of $\frac{\partial(\beta F)}{\partial\beta} = E$:

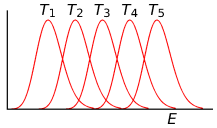
$$\frac{\partial(\beta F)}{\partial\beta} = \frac{-\partial\ln Z}{\partial\beta} = -\frac{1}{Z}\frac{\partial Z}{\partial\beta} = -\frac{\partial\sum_\psi \mathrm{e}^{-\beta\mathcal{E}(\psi)}/\partial\beta}{\sum_\psi \mathrm{e}^{-\beta\mathcal{E}(\psi)}} = -\frac{\sum_\psi\left[-\mathcal{E}(\psi)\mathrm{e}^{-\beta\mathcal{E}(\psi)}\right]}{\sum_\psi \mathrm{e}^{-\beta\mathcal{E}(\psi)}} = \langle\mathcal{E}\rangle = E$$

## Multiple histogram reweighting I



aka WHAM, weighted histogram analysis method

Building the **density of states** as a funtion of energy in a wide range of temperatures from overlapping histograms of energies obtained in a number of simulations at different temperatures (can be extended to other coupling parameter).




$$\pi(E) = D(E)\mathrm{e}^{-\beta E}$$

$$D(E) \propto E^{\text{const}\,N}$$

Configurational integral and residual Helmholtz energy at temperature $T_i$, $\beta_i = 1/k_{\text{B}}T_i$:

$$Q_i = \mathrm{e}^{-\beta_i F_i} = \int \mathrm{e}^{-\beta_i U(\vec{r}^N)}\mathrm{d}\vec{r}^N = \int D(E)\mathrm{e}^{-\beta_i E}\mathrm{d}E$$

where $D(E)$ (aka $W$, $\Omega$) is the density of states:

$$D(E) = \int_{U(\vec{r}^N)=E} 1\,\mathrm{d}\vec{r}^N \equiv \int \delta(U(\vec{r}^N) - E)\,\mathrm{d}\vec{r}^N$$

$$\overset{\text{rectangle}}{\approx} \int_{U(\vec{r}^N)\in(E-\Delta E/2, E+\Delta E/2)} 1\mathrm{d}\vec{r}^N$$

[Ghoufi et al. (2008)]

## Integration over a coupling parameter



Let us consider any dependence $(\beta U)(\lambda)$, e.g.:

$$(\beta U)(\lambda) = \begin{cases} \beta[U_0 + \lambda(U_1 - U_0)] & \lambda = \text{coupling parameter},\ \lambda \in [0,1] \\ \lambda U & \lambda \equiv \beta\text{: see previous slide} \end{cases}$$

then

$$\frac{\partial\beta F_{\text{res}}}{\partial\lambda} = -\frac{\partial\ln Q}{\partial\lambda} = -\frac{1}{Q}\int\frac{\partial\mathrm{e}^{-\beta U}}{\partial\lambda}\mathrm{d}\vec{r}^N = \frac{1}{Q}\int\frac{\partial(\beta U)(\lambda)}{\partial\lambda}\mathrm{e}^{-\beta U(\lambda)}\mathrm{d}\vec{r}^N = \left\langle\frac{\partial(\beta U)(\lambda)}{\partial\lambda}\right\rangle_\lambda$$

$$(\beta F_{\text{res}})(\lambda_1) = (\beta F_{\text{res}})(\lambda_0) + \int_{\lambda_0}^{\lambda_1}\left\langle\frac{\partial(\beta U)(\lambda)}{\partial\lambda}\right\rangle_\lambda \mathrm{d}\lambda$$

where $\langle\cdot\rangle_\lambda$ = mean value in the ensemble (simulation) with potential $U(\lambda)$

**How to integrate:** (over a real variable or coupling parameter)

- Several discrete values of $\lambda_i$:
  - fit to a suitable function and integrate it
  - use a numerical quadrature method; e.g., the Simpson formula
    (points with a higher weight should be simulated longer)
- A little change of $\lambda$ in every MD step (the system is almost in equilibrium) + integration

## Detour: Density of states for a particle in a box



One particle in 1D in a box of length $L$, energies of eigenstates according to the Schrödinger equation:

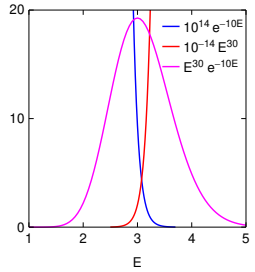
$$E_n = \frac{n^2h^2}{8L^2} \propto n^2, \quad n = 1, 2, \ldots$$

Let's have $f = 3N$ such degrees of freedom. The number of states $\#(E)$ with energy less than $E$ satisfies the equation

$$n_1^2 + n_2^2 + \cdots + n_f^2 < E = [E^{1/2}]^2$$

For large $E$, $\#(E) = 1/2^f$ of the volume of the $E^{1/2}$-ball in the $f$-dimendional space:

$$\#(E) = \frac{\pi^{f/2}E^{f/2}}{\Gamma(f/2+1)} \propto E^{f/2} \quad\Rightarrow\quad D(E) = \frac{\mathrm{d}\#(E)}{\mathrm{d}E} \propto E^{f/2-1}$$




$$\pi(E) = D(E)\mathrm{e}^{-\beta E}$$

The Boltzmann factor $\mathrm{e}^{-\beta E}$ eventually wins!
For large $N$, the product converges to a Gaussian with $\text{Var}(E) \propto f \propto N$ ($\sigma = N^{1/2}$)

## Multiple histogram reweighting II



In the simulation, we calculate the histogram $h_i(E)$ for a set of (equidistant) energies $E$, or some equivalent Gaussian-based $\delta$-function approximation. We will use the normalized histogram and the $\int$-form. (To repeat, subscript $_i$ refers to $T_i$.)

$$\sum_E h_i(E) = \int h_i(E)\mathrm{d}E = 1$$

$$h_i(E) = \frac{D(E)\mathrm{e}^{-\beta_i E}}{\int D(E)\mathrm{e}^{-\beta_i E}\mathrm{d}E} = D(E)\mathrm{e}^{-\beta_i(E-F_i)}$$

Using one temperature only but $F_i$ is not known (yet):

$$D(E) = h_i(E)\mathrm{e}^{\beta_i(E-F_i)} \qquad (2)$$

- We will average $D(E)$ from several simulations at different temperatures.
- $D(E)$ does not depend on $T_i$, but our calculation does ± statistical errors.
- At different $T_i$ different ranges of $E$ are sampled.

⇒ We compose the total $D(E)$ from all data:

$$D(E) = \sum_i w_i(E)h_i(E)\mathrm{e}^{\beta_i(E-F_i)}, \quad \sum_i w_i(E) = 1$$

## Multiple histogram reweighting III



Determining the weights: minimization of the error $\delta D(E)$ (or some estimate). Using reasonable assumptions, we get

$$w_i(E) = \frac{N_i h_i(E)}{\sum_j N_j h_j(E)} = \frac{N_i \mathrm{e}^{-\beta_i(E-F_i)}}{\sum_j N_j \mathrm{e}^{-\beta_j(E-F_j)}}$$

where $N_i$ is the number of measurements at temperature $\beta_i$. ⇒

$$\mathrm{e}^{-\beta_i F_i} = \int D(E)\mathrm{e}^{-\beta_i E}\,\mathrm{d}E = \int \sum_j w_j(E)h_j(E)\mathrm{e}^{\beta_j(E-F_j)}\mathrm{e}^{-\beta_i E}\,\mathrm{d}E$$

$$= \int \frac{\sum_j N_j \mathrm{e}^{-\beta_j(E-F_j)}h_j(E)\mathrm{e}^{\beta_j(E-F_j)}}{\sum_j N_j \mathrm{e}^{-\beta_j(E-F_j)}}\mathrm{e}^{-\beta_i E}\,\mathrm{d}E = \int \frac{\sum_j N_j h_j(E)\mathrm{e}^{-\beta_j E}}{\sum_j N_j \mathrm{e}^{-\beta_j(E-F_j)}}\mathrm{d}E$$

can be solved by iterations (self-consistent solution).
$F_i$ are determined but an additive constant, $D(E)$ but a multiplicative factor

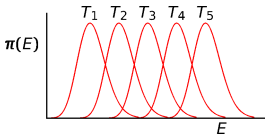
## Multiple histogram reweighting IV



Expectation value at temperature $\beta$:

$$\langle X\rangle_\beta = \frac{\int X(E)D(E)\mathrm{e}^{-\beta E}\mathrm{d}E}{\int D(E)\mathrm{e}^{-\beta E}\mathrm{d}E} = \frac{\int X(E)\dfrac{\sum_i h_i(E)}{\sum_j N_j \mathrm{e}^{-\beta_j(E-F_j)}}\mathrm{e}^{-\beta E}\mathrm{d}E}{\int \dfrac{\sum_i h_i(E)}{\sum_j N_j \mathrm{e}^{-\beta_j(E-F_j)}}\mathrm{e}^{-\beta E}\mathrm{d}E}$$

- $\int \mathrm{d}E$ is over histograms of width $\Delta E$
- if $\Delta E$ is very short, all calculated energies are stored and $\int X(E)h_i(E)\mathrm{d}E$ is replaced by a sum of $E$ over $h_i(E) = 1/N_i$
- $\sqrt{\mathrm{Var}E}/\langle E\rangle \approx N^{-1/2} \Rightarrow \beta_i/\beta_{i+1} \approx 1 \pm N^{-1/2}$ (overlapping distributions)




## Parallel tempering (replica exchange)



$k$ simulations at temperatures $\beta_1 < \beta_2 \ldots$ are run in parallel.

Once a while, 2 subsystems $\beta_i, \beta_j$ (normally $|i-j| = 1$) are interchanged, the acceptance probability is

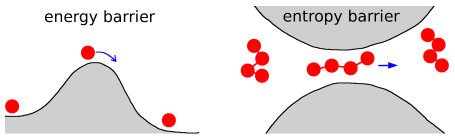
$$\min\left\{1, \frac{\exp(-\beta_i E_j - \beta_j E_i)}{\exp(-\beta_i E_i - \beta_j E_j)}\right\}$$

- Pros: easier barrier crossing, improved ergodicity, faster convergence at low temperatures
- Cons: correlations between subsystems, difficult error estimation

## Conformational flooding, Wang–Landau, metadynamics



Method suitable for fast **barrier crossing** (bad ergodicity – "bottleneck"), incl. calculation of the free energy profile ($\Delta F$ or $\Delta G$), based on decreasing the energy barrier.

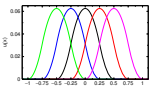



- (several predecessors)
- Conformational flooding H. Grubmüller (1995)
- Wang–Landau (MC) F. Wang, D.P. Landau (2001)
- Metadynamics (MD) A. Laio, M. Parrinello (2002)
- Formal equivalence C. Junghans, D. Perez, T. Vogel (2014)

## Conformational flooding, Wang–Landau, metadynamics



- Our system is defined by the potential $U_0(\vec{r}^N)$
- **Collective variable** $\lambda = \lambda(\vec{r}^N)$ describes the studied process, e.g.: reaction coordinate, atom–atom distance, group–group distance (of their centers of mass e.g., ligand–receptor). various projections or angles,...
  There may be 2 or even more collective variables
- We simulate system with potential $U(\vec{r}^N) = U_0(\vec{r}^N) + \Delta U(\lambda)$, where $\Delta U = 0$ at start
- $\Delta U(\lambda)$ is periodically updated: $\Delta U := \Delta U + \omega\dfrac{\delta(\lambda)}{h(\lambda)}$, $\lambda = \lambda(\vec{r}^N)$
  $\delta$ = aproximation of $\delta$-function (MC: histogram bin, better and MD: Gauss)
  $\omega$ = small enough relaxation parameter, $\omega \ll k_\mathrm{B}T$
  $h(\lambda)$ = density of Cartesian points on hypersurface $\lambda$; e.g., $h(\lambda) = 4\pi r_{12}^2$ for $\lambda = r_{12}$
  ⇒ the probability of visiting the same $\lambda$ again decreases



- If $h(\lambda)$ is omitted in the update stage, it has to be included later.
- Other weight functions may more eficient (= sample different ranges of $\lambda$ better) than $h(\lambda)$; e.g., $h(\lambda) = r_{12}$ for $\lambda = r_{12}$. Again, the true density of points must be taken into account later.

## Conformational flooding, Wang–Landau, metadynamics



**Free energy profile** along $\lambda$:

- strictly, updating should be turned off in the final run (MC: microreversibility violation, MD: heating)
- then, the residual almost-uniform $p(\lambda)$ is determined and:

$$F(\lambda) = \mathrm{const} - \Delta U(\lambda) - k_\mathrm{B}T\ln[p(\lambda)/h(\lambda)]$$

- in practice, with small enough $\omega$ and continued updating, we can assume $p(\lambda)/h(\lambda) = \mathrm{const}^\dagger$ so that (in the sampled region of $\lambda$)

$$F(\lambda) = \mathrm{const} - \Delta U(\lambda)$$

**Free energy** of a well-define "basin" of states (e.g., a bond state) is:

$$F = -k_\mathrm{B}T\ln\int_{\lambda_1}^{\lambda_2}\mathrm{e}^{-F(\lambda)/k_\mathrm{B}T}h(\lambda)\mathrm{d}\lambda$$

integration over region for more collective variables

Two case studies:
- simple MC (next 2 pages)
- potential of mean force in simen08

$^\dagger$symbols const are different

## Conformational flooding, Wang–Landau, metadynamics: Case study

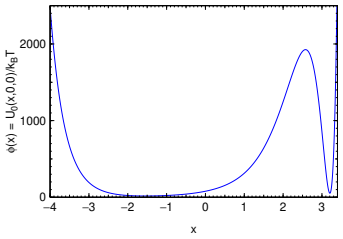
plot/metadynamics.sh 1 2



3D system, interaction energy: online simulation is for T=1 K and T=2 K, type r to reread

$$\frac{U_0(x,y,z)}{k_\mathrm{B}\,\mathrm{K}} = \phi(x)(y^2+z^2+1), \text{ where } \phi(x) = \Big[(x+1)(x+2)(x-2)(x-4)+21\Big]\mathrm{e}^{(x+1.5)^2/3}$$

- The barrier is $E^*/k_\mathrm{B} = 1910\,\mathrm{K}$, $\mathrm{e}^{-E^*/k_\mathrm{B}1\mathrm{K}} = 10^{-929}$
- Collective variable $\lambda = x$
- Histogram: triangular $\delta(\lambda)$, grid = 1/100
- On purpose short MC step in $x$ (≈ MD)
- Initial $\omega = 0.25k_\mathrm{B}T$, decrease to finish
  fine + stop ($\omega \equiv a < 0$): aaaV in the plot window
- graphs shown:
  – running: $\Delta U - F_\mathrm{exact}$, where
  $$F_\mathrm{exact}(\lambda) = \phi(\lambda) + k_\mathrm{B}T\ln(\phi(\lambda))$$
  – final: $\Delta U$ and $\phi(x) = U_0(x,0,0)$
  – $F$ and $F_\mathrm{exact}$ comparison




Note: in real systems, the second term will be a result of many degrees of freedom, not just 2 ($y$, $z$)

## Conformational flooding, Wang–Landau, metadynamics: Case study

xmaple ../maple/metadynamics-case-study.mw

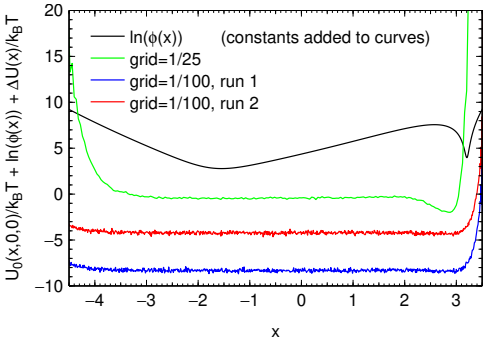


The final graph shows $F(x) + \Delta U(x)$ (two independent simulations $T = 1\,\mathrm{K}$, different start)

| 1/grid | $\Delta F/k_\mathrm{B}T$ |
|---|---|
| 50 | 37.85(8) |
| 100 | 40.52(5) |
| 200 | 40.98(5) |
| 400 | 41.08(3) |
| exact* | 41.13 |

where $\Delta F$ is the difference between the right and left basins.

* see Maple




## Widom particle insertion method I



Open system

$$\mathrm{d}F = -S\mathrm{d}T - p\mathrm{d}V + \mu\mathrm{d}N$$

$$\beta\mu = \left(\frac{\partial(\beta F)}{\partial N}\right)_{V,T} = -\left(\frac{\partial \ln Z_N}{\partial N}\right)_{V,T}$$

$$\beta\mu_\mathrm{res} = \left(\frac{\partial(\beta F_\mathrm{res})}{\partial N}\right)_{V,T} = -\left(\frac{\partial \ln(Q_N/V^N)}{\partial N}\right)_{V,T} \approx -\left(\ln\frac{Q_{N+1}}{V^{N+1}} - \ln\frac{Q_N}{V^N}\right)$$

$$\exp(-\beta\mu_\mathrm{res}) = \frac{1}{V}\frac{Q_{N+1}}{Q_N}$$

Or for the full chemical potential:

$$\mathrm{e}^{-\beta\mu} = \frac{Z_{N+1}}{Z_N} = \frac{Q_{N+1}/[(N+1)!\Lambda^{3(N+1)}]}{Q_N/[N!\Lambda^{3N}]} = \frac{1}{(N+1)\Lambda^3}\frac{Q_{N+1}}{Q_N} \approx \frac{1}{N\Lambda^3}\frac{Q_{N+1}}{Q_N}$$

then by subtracting $\mu^\mathrm{id} = k_\mathrm{B}T\ln\left(\frac{N\Lambda^3}{V}\right)$ we get the same $\mu_\mathrm{res} = \mu - \mu^\mathrm{id}$

## Widom particle insertion method II

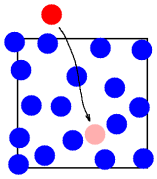


$$\exp(-\beta\mu_{\text{res}}) = \frac{1}{V}\frac{Q_{N+1}}{Q_N}$$

$$N \to N+1 \qquad U_{N+1} = U_N + \Psi(N)$$

$$\frac{1}{V}\frac{Q_{N+1}}{Q_N} = \frac{1}{VQ_N}\int \exp(-\beta U_N - \beta\Psi)\mathrm{d}\vec{r}_1 \ldots \mathrm{d}\vec{r}_{N+1} = \frac{1}{V}\int \langle \mathrm{e}^{-\beta\Psi}\rangle_N \mathrm{d}\vec{r}_{N+1}$$

$$\exp(-\beta\mu_{\text{res}}) = \frac{1}{V}\int \langle \mathrm{e}^{-\beta\Psi}\rangle_N \mathrm{d}\vec{r}_{N+1} = \left\langle \langle \mathrm{e}^{-\beta\Psi}\rangle_N \right\rangle_{\text{random } \vec{r}_{N+1}}$$



where $\frac{1}{V}\int X \mathrm{d}\vec{r}_{N+1} = \langle X\rangle_{\text{random } \vec{r}_{N+1}}$ = mean value of $X$ over positions of the $(N+1)$-th particle in volume $V$, calculated by MC integration (inserting a particle at rancom place)

- $(N+1)$-th particle does not influence the system – it is virtual (fictitious, ghost)
- Problem: dense systems, large solutes
  Remedy: gradual insertion (thermodynamic integration or by finite steps)

**Similar:** solute insertion ⇒ solubility, Henry constant

## Example

cd ../maple; xmaple simul07+18+Widom.mw



We have simulated $N = 500$ Ar atoms (Lennard-Jones: $\sigma = 3.405\,\text{Å}$, $\epsilon/k_\text{B} = 119.8$ K) in a box of volume $V = 15.791\,\text{nm}^3$ at temperature $T = 150$ K. By the Widom method, we found that[§]

$$\exp(-\mu_{\text{res}}/k_\text{B}T) = 0.749(3)$$

Calculate $\mu°$, the chemical potential of Ar with respect to the standard state of ideal gas at pressure $p^{\text{st}} = 1$ bar and temperature $T$.

$\mu^{\text{id}} = k_\text{B}T \ln\frac{N\Lambda^3}{V}$

**Hints:**

$$\begin{aligned}
\mu_{\text{res}} &= \mu - \mu^{\text{id}}(T,V) \quad (N \text{ particles in volume } V)\\
\mu° &= \mu - \mu^{\text{id}}(T,V^{\text{id}}) \quad (V^{\text{id}} = \text{volume of ideal gas of } N \text{ particles at } T, p^{\text{st}})\\
\mu° &= \mu_{\text{res}} + \mu^{\text{id}}(T,V) - \mu^{\text{id}}(T,V^{\text{id}})\\
&= \mu_{\text{res}} + k_\text{B}T \ln\frac{Nk_\text{B}T}{p^{\text{st}}V}
\end{aligned}$$

$$\begin{aligned}
\mu &= 8.449(5)\,\text{kJ mol}^{-1} \text{ (per mole)}\\
\mu &= \mu_{\text{res}} + \Delta\mu = 1.4030(8)\cdot10^{-20}\,\text{J}\\
&= -k_\text{B}T\ln(15.791/10355) = 1.3432\cdot10^{-20}\,\text{J}\\
\Delta\mu &\equiv \mu(15.791\,\text{nm}^3) - \mu(10355\,\text{nm}^3) = -T\Delta S_m\\
V^{\text{id}} &= Nk_\text{B}T/p^{\text{st}} = 10355\,\text{nm}^3\\
\sigma(\mu_{\text{res}}) &= k_\text{B}T \times (0.003/0.749) = 8.3\cdot10^{-24}\,\text{J (std. error)}\\
\mu_{\text{res}} &= 5.985\cdot10^{-22}\,\text{J (per molecule)}
\end{aligned}$$

[§]The value in parentheses is the estimated standard uncertainty in the unit of the least significant digit.

## Henry constant



Gas (2) dissolved in liquid (1), mole fraction in liquid $x_2 = N_2/(N_1+N_2)$, $x_2 \ll 1$.
One form of the Henry law for partial pressure $p_2$ of (2) in equilibrium with solution:

$$p_2 = K_\text{H} x_2$$

By virtual insertion of one molecule ($N_2 = 1$) of gas (2) to pure liquid (1), we get the residual chemical potential of (2) at $x_2 = 1/(N_1+1) \approx 1/N_1$,

$$\left.\begin{aligned}
&\mu_{\text{res},2} = \mu_2(x_2) - \mu_2^{\text{id}}(V)\\
&\text{In equilibrium:}\\
&\mu_2(x_2) \overset{\text{eql.}}{=} \mu^{\text{id}}(V_2) = k_\text{B}T\ln\frac{\Lambda_2^3}{V_2}
\end{aligned}\right\} \mu_{\text{res},2} = k_\text{B}T\ln\frac{N_2\Lambda^3}{V_2} - k_\text{B}T\ln\frac{N_2\Lambda^3}{V} = k_\text{B}T\ln\frac{V}{V_2}$$

$\mu^{\text{id}}(V) = k_\text{B}T\ln\frac{N\Lambda^3}{V}$

where $V_2$ is the volume corresponding to pressure $p_2$ (don't forget that $N_2 = 1$),

$$V_2 = \frac{1k_\text{B}T}{p_2} = \frac{k_\text{B}T}{x_2K_\text{H}} = \frac{k_\text{B}TN_1}{K_\text{H}}$$

where $\rho_1 = N_1/V$ = number density of liquid (1). Finally,

$$K_\text{H} = \frac{\rho_1 k_\text{B}T}{\mathrm{e}^{-\beta\mu_{\text{res},2}}} = \frac{\rho_1 k_\text{B}T}{\langle(\mathrm{e}^{-\beta\Psi})_{N_1}\rangle_{\text{random insertion of (2)}}}$$

## Reversible work by integrating the mean force



From thermodynamics:

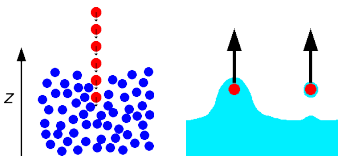
$$\Delta G = W_{\text{other than pressure–volume}} \quad [p,T]$$

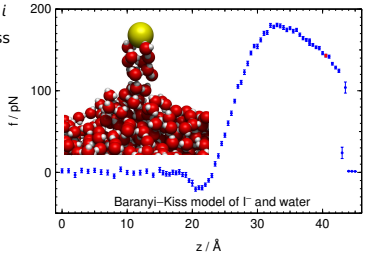
$$\Delta\mu_i = -\int_{\vec{r}_i(1)}^{\vec{r}_i(2)} \langle\vec{f}_i\rangle \cdot \mathrm{d}\vec{r}_i$$

where $\vec{f}_i = -\partial U/\partial\vec{r}_i$ is the force acting on particle $i$

Molecules: the force applies to the center of mass or other reference point.

Problem: hysteresis




## Mean force and its potential



Let us define the **single-particle density** as (also denoted as $n_1$)

$$\rho_1(\vec{r}_1) = \frac{N}{Q_N}\int \mathrm{e}^{-\beta U(\vec{r}_1,\ldots,\vec{r}_N)}\mathrm{d}\vec{r}_2 \ldots \mathrm{d}\vec{r}_N$$

– $\rho_1(\vec{r}_1)\mathrm{d}\vec{r}_1$ is the probability of finding a particle (any one) in $\mathrm{d}\vec{r}_1$
– extension to mixtures: $N/Q_N \to N_{\text{species}}/Q_N$

The **potential of mean force** is defined by

$$\overline{U}_1(\vec{r}_1) = -k_\text{B}T\ln[V\rho_1(\vec{r}_1)]$$

The corresponding force is

$$\bar{f}_i = -\left(\frac{\partial\overline{U}_1}{\partial\vec{r}_1}\right) = k_\text{B}T\frac{\partial\rho_1/\partial\vec{r}_1}{\rho_1} = \left\langle -\left(\frac{\partial U}{\partial\vec{r}_1}\right)\right\rangle_{\vec{r}_2,\ldots,\vec{r}_N} = (\vec{f}_1)_{\vec{r}_2,\ldots,\vec{r}_N}$$

i.e,. it is indeed the mean force on particle 1 held at fixed position $\vec{r}_1$.

Note: Similarly, for a 2-particle distribution, which for a pair of particles in an isotropic fluid is $g(r)$, one defines the potential of mean force as:

$$\overline{U}_2(r) = -k_\text{B}T\ln[g(r)]$$

## Interfacial (surface) energy of solids



**Cleaving** [Davidchack, Laird: JCP 118, 7651 (2003)]: Thermodynamic integration over a gradually growing "knife" (e.g., Gaussian potential) inserted between crystalographic planes.

**Molding** [Espinosa, Vega, Sanz: JCP 141, 134709 (2014)]: Thermodynamic integration over a gradually growing "mold" (potential wells) to keep a crystal growing in a part of the system.

- General problem of both methods: hysteresis

## Gibbs energy of crystals



- Einstein crystal[¶] at given $T$ as a reference, integration over a coupling parameter [Frenkel, Ladd: JCP 81 3188 (1984), Frenkel, Mulder: Mol. Phys. 55, 1171 (1985)].
- Classical method: harmonic vibrations as a reference, thermodynamic integration of the difference $0 \to T$ [Kolafa JCTC 15, 68 (2019) and references therein]

[¶]independent harmonic oscillators, here treated classically

## Local density/concentration method



Let a solute $i$ be a subject of external potential $U_i^{\text{ext}}(\vec{r})$ (e.g., "gravity"). In equilibrium:

$$\mu_i(\vec{r}) + U_i^{\text{ext}}(\vec{r}) = \text{const}$$

or

$$\mu_i(\vec{r}_1) - \mu_i(\vec{r}_2) = -[U_i^{\text{ext}}(\vec{r}_1) - U_i^{\text{ext}}(\vec{r}_2)]$$

We determine the concentration or density at $\vec{r}_1$ and know the chemical potential (with respect to a certain reference)

**Example**:

- reference = infinite dilution approximation ($\gamma = 1$) in the region of small concentration
- the activity coefficient $\gamma$ in the region of high concentration can be calculated