Compressibility and fluctuations I

Identical particles in the grand canonical ensemble (μVT), semiclassical approximation, $G = \mu N$:

$$\Omega = F - \mu N = F - G = -pV = -k_\mathrm{B} T \ln Z_{\mu VT}, \quad \text{kde } Z_{\mu VT} = \sum_{N=0}^{\infty} \frac{\mathrm{e}^{\beta\mu N}}{N! h^{3N}} \int \mathrm{e}^{-\beta E} \mathrm{d}\vec{r}_1 \ldots\ldots \mathrm{d}\vec{p}_N$$

Let's temporarily distinguish $\langle N \rangle$ and the summation index N

$$\langle N \rangle = -\left(\frac{\partial \Omega}{\partial \mu}\right)_{V,T} = k_\mathrm{B} T \left(\frac{\partial \ln Z_{\mu VT}}{\partial \mu}\right)_{V,T} = \frac{\sum_0^\infty N \frac{\mathrm{e}^{\beta\mu N}}{N! h^{3N}} \int \mathrm{e}^{-\beta E} \mathrm{d}\vec{r}_1 \ldots \mathrm{d}\vec{p}_N}{\sum_0^\infty \frac{\mathrm{e}^{\beta\mu N}}{N! h^{3N}} \int \mathrm{e}^{-\beta E} \mathrm{d}\vec{r}_1 \ldots \mathrm{d}\vec{p}_N} = \frac{\sum_{N=0}^\infty N \frac{\mathrm{e}^{\beta\mu N}}{N! \Lambda^{3N}} Q_N}{\sum_{N=0}^\infty \frac{\mathrm{e}^{\beta\mu N}}{N! \Lambda^{3N}} Q_N},$$

$$\begin{aligned}\left(\frac{\partial \langle N \rangle}{\partial \mu}\right)_{V,T} &= \beta \frac{\sum_{N=0}^\infty N^2 \frac{\mathrm{e}^{\beta\mu N}}{N! \Lambda^{3N}} Q_N \times \sum_{N=0}^\infty \frac{\mathrm{e}^{\beta\mu N}}{N! \Lambda^{3N}} Q_N - \sum_{N=0}^\infty N \frac{\mathrm{e}^{\beta\mu N}}{N! \Lambda^{3N}} Q_N \times \sum_{N=0}^\infty N \frac{\mathrm{e}^{\beta\mu N}}{N! \Lambda^{3N}} Q_N}{\left(\sum_{N=0}^\infty \frac{\mathrm{e}^{\beta\mu N}}{N! \Lambda^{3N}} Q_N\right)^2} \\ &= \beta \left(\langle N^2 \rangle - \langle N \rangle^2\right) = \beta \langle (N - \langle N \rangle)^2 \rangle = \beta \operatorname{Var} N\end{aligned}$$

$$\mathrm{d}F = -S\mathrm{d}T - p\mathrm{d}V + \mu\mathrm{d}N$$
$$\mathrm{d}\Omega = -S\mathrm{d}T - p\mathrm{d}V - N\mathrm{d}\mu$$

Compressibility and fluctuations II

$$dF = -S dT - p dV + \mu dN$$
$$dG = -S dT + V dP + \mu dN$$
$$d\Omega = -S dT - p dV - N d\mu$$

For constant $[T, N]$ it holds:

$$dG = V dP, \quad \text{but also} \quad dG = d(\mu N) = N d\mu$$

Alternatively:

$$d\Omega = d(-pV) = -p dV - V dp = \overset{=0 \text{ for } [T]}{-S dT} - p dV - N d\mu \quad \Rightarrow \quad N d\mu = V dP$$

Let's write

$$N \left(\frac{\partial \mu}{\partial N} \right)_{T,V} = \frac{N}{\beta \operatorname{Var} N}$$

p and μ are intensive variables' therefore, they depend on $\rho = N/V$ (here, $N = \langle N \rangle$):

$$N \left(\frac{\partial \mu}{\partial N} \right)_{T,V} = V \left(\frac{\partial p}{\partial N} \right)_{T,V} = \left(\frac{\partial p}{\partial (N/V)} \right)_T = \frac{1}{N} \left(\frac{\partial p}{\partial (1/V)} \right)_{T,N} = \frac{V}{N} \left[-V \left(\frac{\partial p}{\partial V} \right)_{T,N} \right] = \frac{1}{\rho \kappa_T}$$

Eventually: $$\frac{\operatorname{Var} N}{\langle N \rangle} = \rho k_B T \kappa_T$$

isothermic compressibility: $\kappa_T = -\frac{1}{V} \left(\frac{\partial V}{\partial p} \right)_{T,N}$

bulk modulus: $B_T = 1/\kappa_T$

Compressibility and fluctuations III

$$\frac{\operatorname{Var} N}{\langle N\rangle} = \rho k_\mathrm{B} T \kappa_T$$

- larger compressibility ⇒ larger fluctuations
- $\operatorname{Var} N > 0 \Rightarrow \kappa_T > 0$ ($\kappa_T < 0$ for a mechanically unstable system)
- $\operatorname{Var} \rho = \dfrac{\rho^3 k_\mathrm{B} T \kappa_T}{N} \overset{N\to\infty}{=} 0$ (thermodynamic limit)

Typical “finite-size effect” is $\mathcal{O}(1/N)$

Exceptions:

- diffusivity in MD: $\mathcal{O}(1/N^{1/3})$ – a particle interacts with its periodic image $\propto 1/N^{1/3}$ apart
- crystals: $\mathcal{O}(\log N/N)$ – counting phonons
- plasma, ionic solutions (more terms): $\mathcal{O}(1/N^{3/2})$ – Debye–Hückel
- some 2D systems: $\mathcal{O}(\log N/N)$
- critical point – critical exponents

RDF in the μVT ensemble and compressibility I

$$\rho^2 g(r) = \frac{\displaystyle\sum_{N=2}^{\infty} N(N-1)\frac{e^{\beta\mu N}}{N!\Lambda^{3N}}\int \exp(-\beta U)d\vec{r}_3\dots d\vec{r}_N}{\displaystyle\sum_{N=0}^{\infty} \frac{e^{\beta\mu N}}{N!\Lambda^{3N}}\int \exp(-\beta U)d\vec{r}_1\dots d\vec{r}_N}, \qquad r = |\vec{r}_1 - \vec{r}_2|$$

⇒ **Compressibility equation** $1 + \rho\int [g(r)-1]\,d\vec{r} = \frac{\mathrm{Var}\,N}{\langle N\rangle} = \rho k_B T \kappa_T$

spherical symmetry:
$\int d\vec{r} = \int_0^\infty 4\pi r^2 dr$

Proof:

$\rho = \langle N\rangle / V$

$$\rho\int g(\vec{r})d\vec{r} = \frac{\rho}{V}\int g(\vec{r}_{12})d\vec{r}_{12}d\vec{r}_1 \overset{J=1}{=} \frac{\rho}{V}\int g(\vec{r}_{12})d\vec{r}_1 d\vec{r}_2 = \frac{\rho}{V}\frac{\langle N(N-1)\rangle}{\rho^2} = \frac{\langle N(N-1)\rangle}{\langle N\rangle}$$

$$\rho\int 1 d\vec{r}_{12} = \frac{\langle N\rangle}{V}V = \langle N\rangle$$

all three terms together:

$$1 + \rho\int [g(r)-1]\,d\vec{r} = \frac{\langle N\rangle + \langle N(N-1)\rangle - \langle N\rangle^2}{\langle N\rangle} = \frac{\langle N^2\rangle - \langle N\rangle^2}{\langle N\rangle} = \frac{\mathrm{Var}\,N}{\langle N\rangle}$$

RDF in the μVT ensemble and compressibility I

- More fluctuation and correlation quantities can be expressed by similar integrals (Kirkwood–Buff)
- Numerically ill-defined for large r – must be cut off
- Tricks to be able to use the NVT ensemble

Exercise 1. Show that $g_{\mu VT}(r) = 1$ for monoatomic ideal gas
Hint: for ideal gas it holds $\mathrm{e}^{\beta\mu N}/\Lambda^3 = p/k_\mathrm{B}T = \langle N\rangle/V = \rho$

Exercise 2. Calculate κ_T from the compressibility equation with the canonical RDF (N = constant)

$\kappa_T = 0$ – incompressible by definition

Simulation in other ensembles

- $NVE \to NVT$ (MD), measuring: $T \to E$
- $NVT \to NVE$ (MC), measuring: $E \to T$
- $NVT \to NPT$ (MC, MD), measuring: $P \to V$
- $NVT \to \mu VT$ (MC, [MD]), measuring: $\mu \to N$

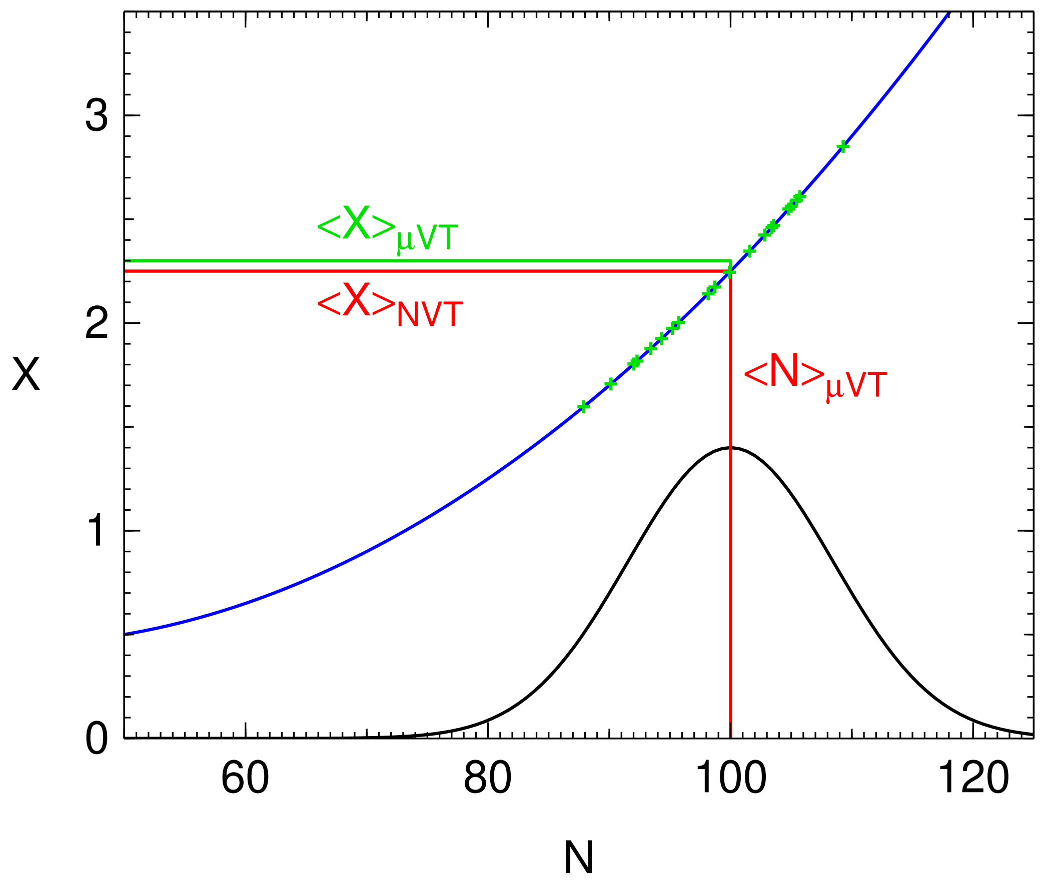
Typical error $\propto 1/N \to 0$ for $N \to \infty$ *:

Corrections:

$$\langle X\rangle_{\mu\mathrm{VT}} - \langle X\rangle_{NVT} \approx \frac{1}{2}\left\langle \left(\overset{N = \langle N\rangle_{\mu\mathrm{VT}}}{N} - \langle N\rangle_{\mu\mathrm{VT}}\right)^2\right\rangle_{\mu\mathrm{VT}} \left(\frac{\partial^2 \langle X\rangle_{\mu\mathrm{VT}}}{\partial N^2}\right)_{V,T}$$

$$= \frac{k_\mathrm{B}T}{2N}\left(\frac{\partial\rho}{\partial p}\right)_T \rho^2 \left(\frac{\partial^2\langle X\rangle}{\partial\rho^2}\right)_T$$

where $\langle\cdot\rangle$ in the last derivative is either $\langle\cdot\rangle_{\mu VT}$ or $\langle\cdot\rangle_{NPT}$



Derivation: Taylor expansion of $X(N)$ around $\langle N\rangle$

The corrections become important near the critical point

* not for: nonperiodic b.c. (surface $N^{2/3}$)
crystals ($\ln N/N$)
diffusivity ($N^{1/3}$)
at critical point (slower)

MC in the microcanonical ensemble

MC move under constraint $E = \text{const}$ = problem

It is possible in the classical mechanics for $E_{\text{pot}} + E_{\text{kin}} = \text{const}$: can be integrated over momenta (not so trivial, though).

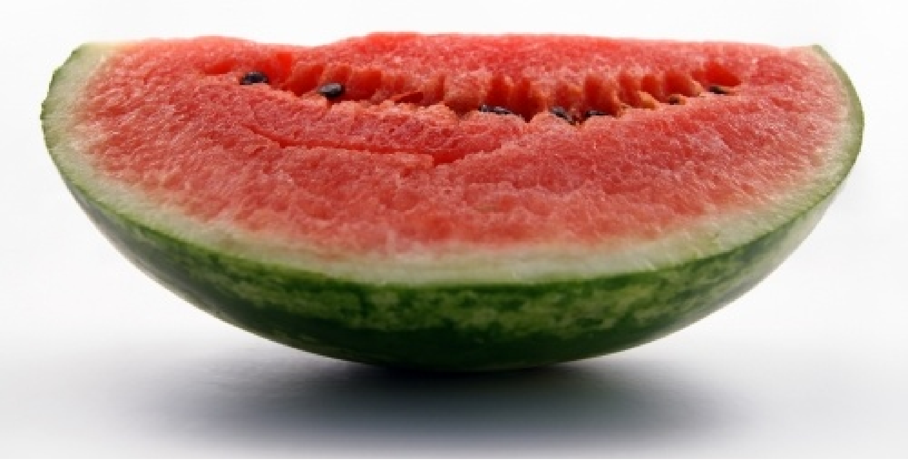
Approximate solution – Creutz

$$E = E_{\max} \quad \rightarrow \quad E \leq E_{\max}$$

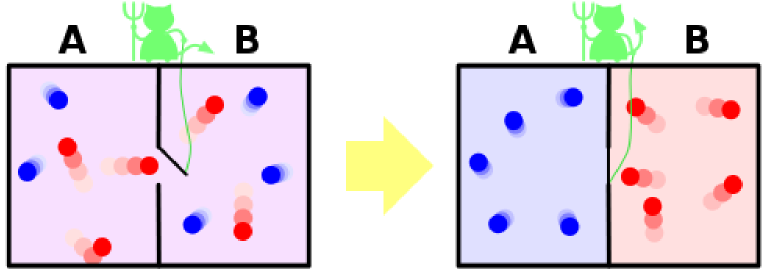
(do not buy a melon in a many-dimensional space)

Creutz demon has a bag with energy: $E_{\text{bag}} = E_{\max} - E \geq 0$

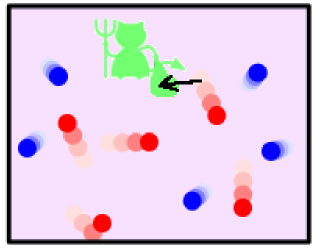
E_{bag} has the Boltzmann distribution ⇒ temperature



Maxwell's demon



Creutz's demon



Credit: Wikipedia (modified)

Creutz – Metropolis comparison

- Choose a particle (lattice site, . . .) to move
- $A^{\rm tr} := A^{(k)}$ + random move of the chosen particle
- $\Delta U := U(A^{\rm tr}) - U(A^{(k)}) \equiv U^{\rm tr} - U^{(k)}$
- The configuration is accepted ($A^{(k+1)} := A^{\rm tr}$) with probability $\min\{1, {\rm e}^{-\beta\Delta U}\}$ otherwise rejected:

Metropolis	Creutz	Creutz–Metropolis
$u := u_{(0,1)}$ IF $u < {\rm e}^{-\beta\Delta U}$ THEN $A^{(k+1)} := A^{\rm tr}$ ELSE $A^{(k+1)} := A^{(k)}$	IF $\Delta U <$ bag THEN $A^{(k+1)} := A^{\rm tr}$; bag $-=\Delta U$ ELSE $A^{(k+1)} := A^{(k)}$	bag $= -k_{\rm B}T \ln u_{(0,1)}$ IF $\Delta U <$ bag THEN $A^{(k+1)} := A^{\rm tr}$; bag $-=\Delta U$ ELSE $A^{(k+1)} := A^{(k)}$

in all cases $\langle {\rm bag}\rangle = k_{\rm B}T$ (in continuous world: $\langle -\ln u_{(0,1)}\rangle = 1$)

- $k := k+1$ and again and again

NPT ensemble in MC

To incorporate volume change, $\langle X\rangle$ must be in the form of an $\int$ of probability density:

$\vec{r}_i = V^{1/3}\vec{\xi}_i$

$$\begin{aligned}\langle X\rangle &= \frac{1}{Q_{\rm NPT}}\int_0^\infty\left(\int_{V^N} X(\vec{r}^N, V)\frac{N}{V}\exp\{-\beta[pV+U(\vec{r}^N)]\}\,\mathrm{d}\vec{r}^N\right)\mathrm{d}V\\ &= \frac{1}{Q_{\rm NPT}}\int_0^\infty\int_{1^{3N}} X(V^{1/3}\vec{\xi}^N, V)\frac{N}{V}V^N\exp\{-\beta[pV+U(V^{1/3}\vec{\xi}^N)]\}\,\mathrm{d}\vec{\xi}^N\mathrm{d}V\end{aligned}$$

N/V, $^{-1}$: exact ideal gas
βp, const: $\mathcal{O}(1/N)$ error

Microreversible volume change: $V^{\rm tr} = V + u_{(-d,d)}$, then

$$p_{\rm acc} = \min\{1, (V^{\rm tr}/V)^{N-1}\exp[-\beta p(V^{\rm tr}-V)]\exp[-\beta(U^{\rm tr}-U)]\}$$

Better option: $V^{\rm tr} = V\exp[u_{(-d,d)}]$ ($\ln V$ is uniformly sampled), then:

$$p_{\rm acc} = \min\{1, (V^{\rm tr}/V)^{N+1-1}\exp[-\beta p(V^{\rm tr}-V)]\exp[-\beta(U^{\rm tr}-U)]\}$$

- Usually N one-particle moves (translations:rotations = 1:1) per one volume-change move
- Acceptance ration of volume changes ~ 0.3
- General problem: global change of configuration ⇒ slow convergence ⇒ not good for too large systems

Grandcanonical ensemble in MC

$\vec{r}_i = V^{1/3}\vec{\xi}_i$

- MC step: change the number of particles by ±1

$$\langle X\rangle = \frac{1}{\Xi}\sum_{N=0}^{\infty}\frac{e^{\beta\mu N}}{\Lambda^{3N}N!}\int X(\vec{r}^N, N)\exp[-\beta U_N(\vec{r}^N)]\,d\vec{r}^N$$

$d\vec{r}^N$ depends on N ⇒ dimensionless coordinates $\vec{r}_i = V^{1/3}\vec{\xi}_i$

$$\langle X\rangle = \frac{1}{\Xi}\sum_{N=0}^{\infty}\int_{1^{3N}} X(V^{1/3}\vec{\xi}^N, N)\frac{e^{\beta\mu N}V^N}{\Lambda^{3N}N!}\exp[-\beta U_N(V^{1/3}\vec{\xi}^N)]\,d\vec{\xi}^N$$

- Insert or remove a particle with the same probability 1/2

$$p_{\text{insert particle}} = \min\left\{1, \frac{e^{\beta\mu}V}{\Lambda^3(N+1)}\exp\{-\beta[U_{N+1}(\vec{r}^{N+1,\text{zkus}}) - U_N(\vec{r}^N)]\}\right\}$$

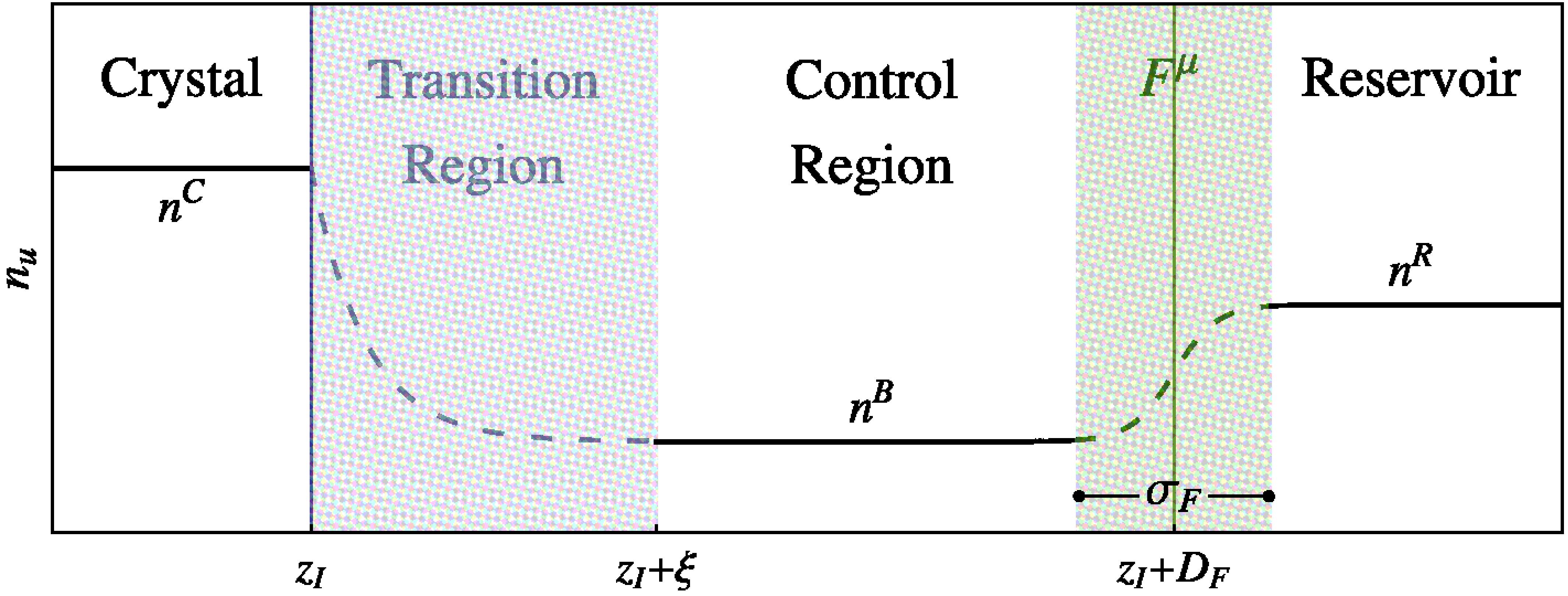
$$p_{\text{remove particle}} = \min\left\{1, \frac{N\Lambda^3}{e^{\beta\mu}V}\exp\{-\beta[U_{N-1}(\vec{r}^{N-1,\text{zkus}}) - U_N(\vec{r}^N)]\}\right\}$$

Problem: insert a large molecule
Solution: gradually

$e^{\beta\mu} = \Lambda^3 e^{\beta\mu_{\text{res}}}\langle N\rangle/V$

Grandcanonical ensemble in MD

- The same as in MD, but "continuously" – problematic
- CμMD [Perego, Salvalaglio, Parrinello, DOI: 10.1063/1.4917200]
  - Reservoir with molecules, region with a force ⇒ change of the (chem.) pot.
  - Applied to crystallization with a constant oversaturation of the solution



Reaction ensemble in MC

- We can easily calculate a chemical equilibrium in an ideal gas phase.
  But what if the mixture is not ideal? 1) Calculate μ_i, γ_i... 2) Reaction ensemble

Reaction (reactants: $\nu_i < 0$, products: $\nu_i > 0$): $\displaystyle\sum_{i=1}^{k} \nu_i A_i = 0$

Equilibrium: $$\Delta_\mathrm{r} G_\mathrm{m} = \sum_{i=1}^{k} \nu_i \mu_i = 0$$

Generalized partition function of a mixture, $N = \sum_{i=1}^{k} N_i$ (constant N_i):

$$Z(N_1, \ldots, N_k, V, T) = \prod_{i=1}^{k} \frac{(q_i/\Lambda_i^3)^{N_i}}{N_i!} \times \int \exp[-\beta U(\vec{r}^N)] \mathrm{d}\vec{r}^N$$

Balance (extent of reaction = ζ): $$N_i = N_i^{(0)} + \zeta \nu_i$$

$$Z(N_1^{(0)}, \ldots, N_k^{(0)}, V, T) = \sum_{\zeta} \prod_{i=1}^{k} \frac{(V q_i/\Lambda_i^3)^{N_i^{(0)} + \zeta\nu_i}}{(N_i^{(0)} + \zeta\nu_i)!} \times \int \exp[-\beta U(V^{1/3} \vec{\xi}^N)] \mathrm{d}\vec{\xi}^N$$

Reaction ensemble in MC

Reaction “move” $\zeta^{\text{tr}} = \zeta + \Delta\zeta$ accepted with probalility

$$p_{\text{acc}} = \min\left\{1,\ K'^{\Delta\zeta}\exp(-\beta\Delta U)\prod_{i=1}^{k}\left[\frac{(N_i^{(0)}+\zeta\nu_i)!}{(N_i^{(0)}+\zeta^{\text{tr}}\nu_i)!}\right]\right\}$$

where

$$\Delta U = U(V^{1/3}\vec{\xi}^N,\zeta^{\text{tr}}) - U(V^{1/3}\vec{\xi}^N,\zeta)$$

$$\bar{\nu} = \sum_{i=1}^{k}\nu_i$$

$$K' = \prod_{i=1}^{k}\left(\frac{Vq_i}{\Lambda_i^3}\right)^{\nu_i} = \left(\frac{Vp}{kT}\right)^{\bar{\nu}}\exp\left(-\frac{\sum\mu_{i,\text{id}}}{kT}\right) = \left(\frac{Vp}{kT}\right)^{\bar{\nu}}K$$

where $\Delta_{\text{r}}G_{\text{m}}^{\circ} = N_{\text{A}}\sum\mu_{i,\text{id}}$ is the reaction molar Gibbs energy (for p = standard pressure) and K is the equilibrium constant (for the standard state ideal gas at pressure p).

Reaction step

- Random change of the extent of the reaction: with probalility 1/2 "→" ($\zeta^{\rm tr} = \zeta + 1$) with probalility 1/2 "←" ($\zeta^{\rm tr} = \zeta - 1$)
- Random selection of the corresponding number of reactant and product molecules
- Replacement of reactants → products (for $\Delta\zeta = \zeta^{\rm tr} - \zeta > 0$) or products → reactants (for $\Delta\zeta < 0$)
- Calculate the energy change ΔU
- New configuration accepted with probability $p_{\rm acc}$

Note: Some degrees of freedom are simulated, some not...

Nonspherical molecules:

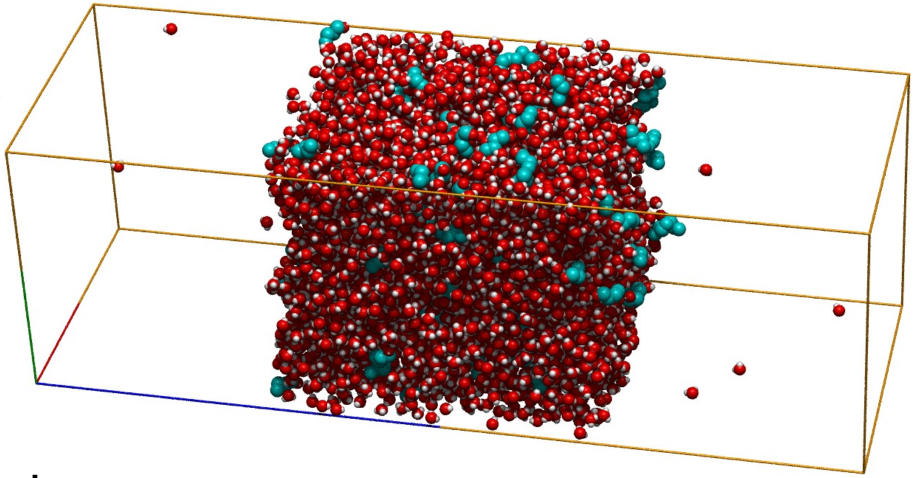
$$\exp\left[\frac{-\mu_{i,\rm id}}{kT}\right] = \frac{q_i k_{\rm B} T}{q_i^{\rm model} p^{\rm st}}, \quad q_i^{\rm model} = \int \exp(-\beta U_{\rm int}) {\rm d(intern.deg.of\ freedom)}$$

Eg., general hard molecule: $q_i^{\rm model} = 8\pi^2 \Rightarrow K'$ must be divided by product $\prod_{i=1}^{k} (q_i^{\rm model})^{\nu_i}$

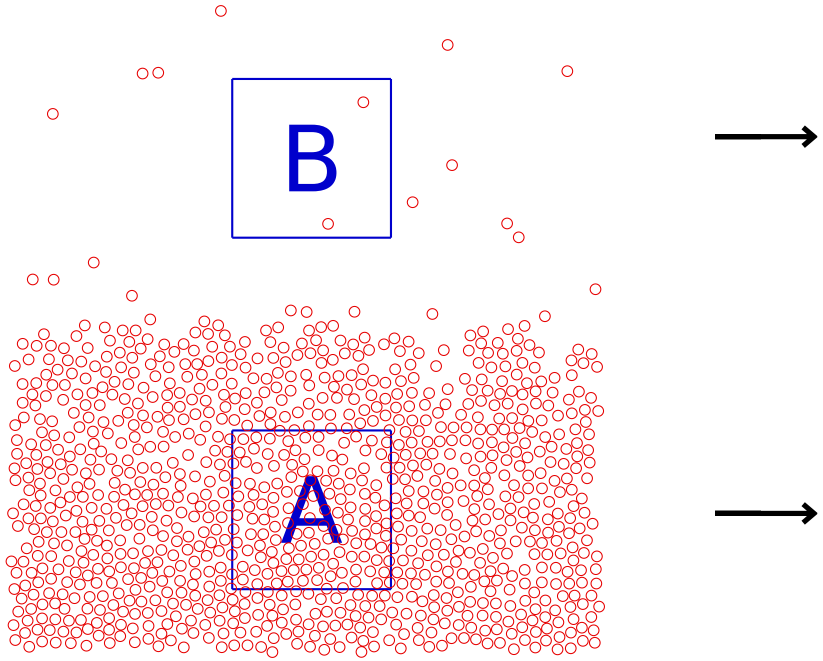
- Again, gradual insertion may be needed
- Final result = equilibrium composition

Gibbs ensemble

Determine vapor–liquid (fluid–fluid) phase equilibrium:
1) MD: slab geometry, bad for low T (water + BuOH, 373 K) →
2) MC, MD: μ in the liquid, μ gas from the virial EoS
3) Gibbs ensemble [A. Panagiotopoulos (1987)]



One-component system:



periodic box

periodic box

- $T = \text{const}$, $V = V_A + V_B = \text{const}$, $N = N_A + N_B = \text{const}$
  ⇒ to be satisfied: $p_A = p_B$ and $\mu_A = \mu_B$
- Gibbs phase law: 1 degree of freedom ⇒ pressure is determined

Gibbs ensemble: one-component system

$$Q_{\rm NVT} = \sum_{N_A=0}^{N} \int_0^V \frac{{\rm d}V_A V_A^{N_A}}{N_A!} \int {\rm d}\vec{\xi}_A^N {\rm e}^{-\beta U_A(N_A)} \frac{V_B^{N_B}}{N_B!} \int {\rm d}\vec{\xi}_B^N {\rm e}^{-\beta U_B(N_B)}$$

- Volume change $V_A^{\rm tr} = V_A + \Delta V$ a $V_B^{\rm tr} = V_B - \Delta V$, acceptance probability:

$$p_{\rm acc} = \min\left\{1, \exp\left[-\beta\Delta U_A - \beta\Delta U_B + N_A \ln\frac{V_A+\Delta V}{V_A} + N_B \ln\frac{V_B-\Delta V}{V_B}\right]\right\}$$

- Particle transfer from box B to box A, acceptance probability:

$$p_{\rm acc} = \min\left\{1, \exp\left[-\beta\Delta U_A - \beta\Delta U_B - \ln\frac{(N_A+1)V_B}{N_B V_A}\right]\right\}$$

- Particle transfer from box A to box B, acceptance probability:

$$p_{\rm acc} = \min\left\{1, \exp\left[-\beta\Delta U_B - \beta\Delta U_A - \ln\frac{(N_B+1)V_A}{N_A V_B}\right]\right\}$$

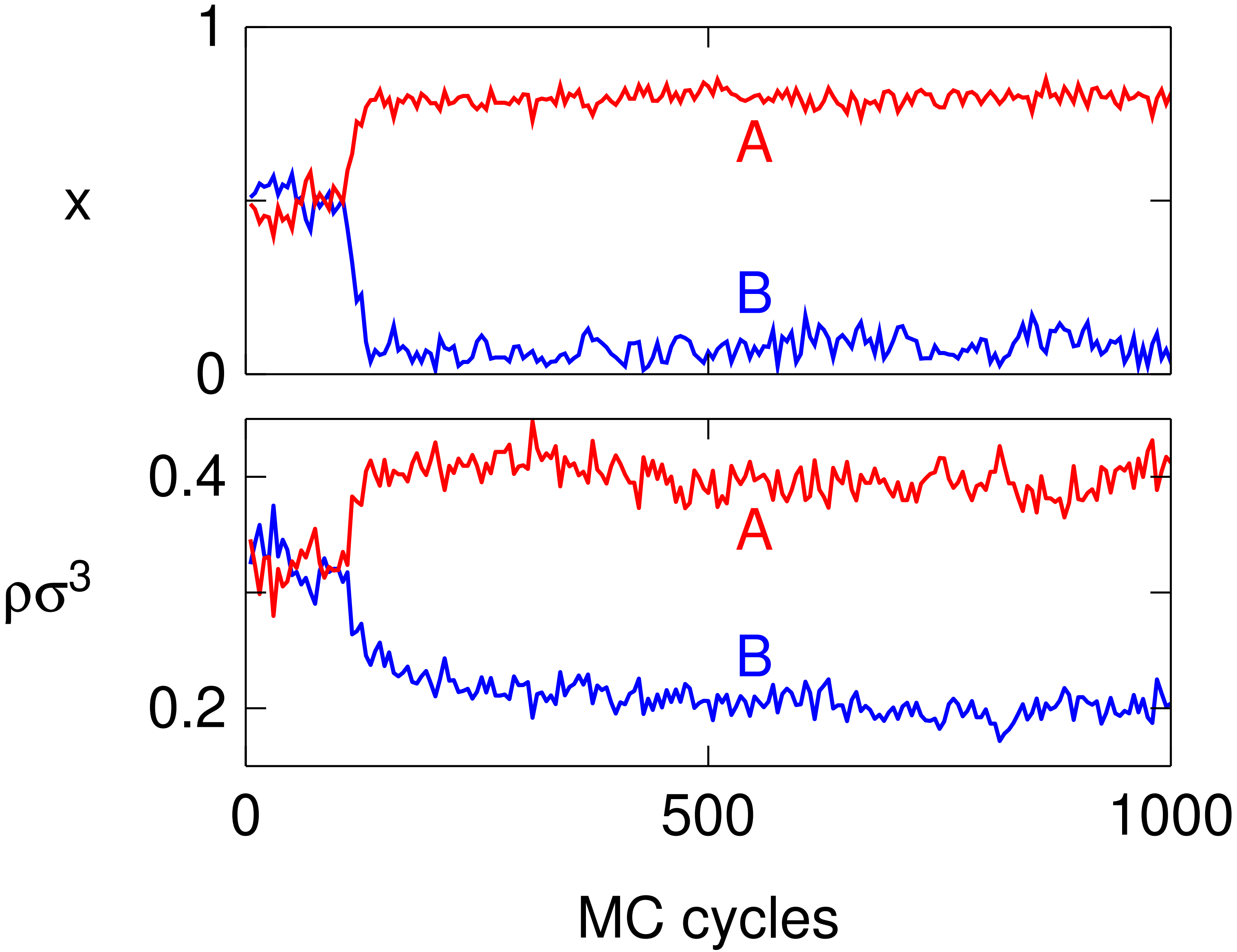
- Standard MC moves – translations, rotations.

Usually 1 volume change + 1–several article transfers per N single-particle moves.

Gibbs ensemble: binary mixture

Gibbs phase law for a binary mixture: 2 degrees of freedom
T = const, p = const, equilibrium compositions are determined

- Volume changes in both boxes separately (see *NPT*)
- Particle transfer
- Useful: particle exchange between boxes – higher probability



credit: Martin Strnad †

Exercise – Madelung constant of NaCl

For the electrostatic energy of a crystal it holds (per 1 molecule of NaCl)

$$E_{\mathrm{el}} = \frac{Ma^2}{4\pi\epsilon_0 a}$$

where a = lattice constant and M = Madelung constant

- Simulate a NaCl crystal while decreasing temperature to zero.
  Use data:

```
T=0 tau.T=0.2
cutoff=8 LJcutoff=cutoff
tau.P=0.2 bulkmodulus=1e11 ;
! restart after cooling
el.epsk=0.1 el.epsr=0.1 el.grid=1024 ! higher precision
init="start" no=1;
```

- Determine the box size and calculate a
- Determine the electrostatic energy and express it per 1 NaCl
- Calculate the Madelung constant

-1.74756