Mathematics for chemical engineers

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3. Implicit functions

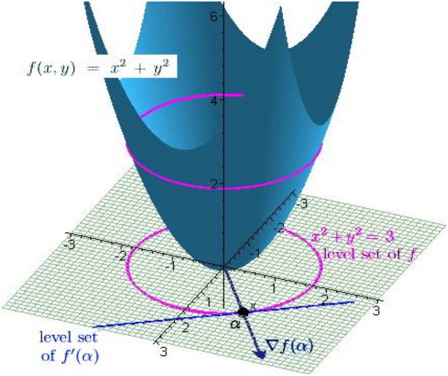
Outline

Implicit functions of one variable

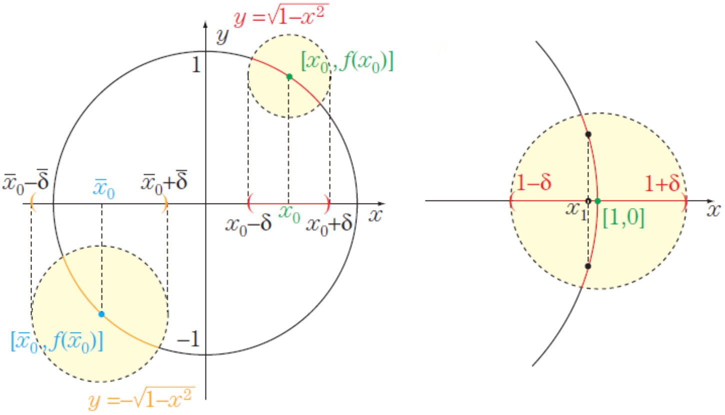
Definition We say that the equation $F(x,y)=0$ defines on the neighborhood of the point (x_0,y_0) implicit function $y=f(x)$, if and only if

1. $F(x_0,y_0)=0$,
2. $\exists\delta>0,\ \epsilon>0\ :\ \forall x\in(x_0-\delta,\ x_0+\delta)$ is $y=f(x)$ the only number in the interval $(y_0-\epsilon,\ y_0+\epsilon)$, that satisfies the equation $F(x,y)=0$.

Remark: $F(x,y)=0\ \ldots$ zero level set of the function of two variables $F=F(x,y)$. In the interval $(x_0-\delta,\ x_0+\delta)$, it is possible to describe this zero level set as a graph of the function $y=f(x)$ (of one variable), where the domain of the definition of the function f is $\mathcal{D}(f)=(x_0-\delta,\ x_0+\delta)$.



Geometrically: The implicitly defined function of one variable $y = g(x)$ describes a part of the intersection of the graph of the function of two variables $\underbrace{z = f(x, y)}_{\text{surface in } \mathbb{R}^3}$ with the plane $z = 0$. Here, $f(x, y) = x^2 + y^2 - 3$.



The equation $F(x, y) = x^2 + y^2 - 1 = 0$ defines implicit function $y = f(x)$ on the neighborhood of points $[x_0, f(x_0)]$, $[\bar{x}_0, f(\bar{x}_0)]$. In the neighborhood of the point $[1, 0]$ the equation doesn't define any implicit function $y = f(x)$.

Theorem: The existence of the implicitly defined function of one variable

Let $F \in \mathrm{C}^k(G)$, $G \subset \mathbb{R}^2$ is an open set, $k \geq 1$. Let $(x_0, y_0) \in G$ be such a point that

1. $F(x_0, y_0) = 0$,
2. $\dfrac{\partial F}{\partial y}(x_0, y_0) \neq 0$.

Then the equation $F(x, y) = 0$ defines in the neighborhood of the point (x_0, y_0) implicitly a function $y = f(x)$ of one variable.

Moreover, $f \in \mathrm{C}^k(x_0 - \delta, x_0 + \delta)$ for a certain $\delta > 0$.

Theorem: The derivative of the implicit function of one variable

Under the assumptions of the existence theorem, the derivative of the implicitly defined function $y = f(x)$ is computed by the formula

$$f'(x) = -\frac{\dfrac{\partial F}{\partial x}(x, f(x))}{\dfrac{\partial F}{\partial y}(x, f(x))} \quad \text{for} \quad x \in (x_0 - \delta, x_0 + \delta). \tag{1}$$

★ Example

Example Calculate y', y'' of the implicit function defined by the equation

$$(x^2 + y^2)^2 - 3x^2y - y^3 = 0 \tag{2}$$

at the neighborhood of the point $(0, 1)$.

Solution At first, let us check the assumptions of the existence of the function $y = f(x)$ defined implicitly by the equation

$F(x, y) \equiv (x^2 + y^2)^2 - 3x^2y - y^3 = 0$ in the neighborhood of $(0, 1)$:

$$F(0, 1) = 0\,,$$

$$\frac{\partial F}{\partial y}(x, y) = 2(x^2 + y^2)2y - 3x^2 - 3y^2\,, \quad \frac{\partial F}{\partial y}(0, 1) = 1 \neq 0\,,$$

i.e., the equation $F(x, y) = 0$ defines on the neighborhood of $(0, 1)$ implicitly a function $y = f(x)$ of one variable. The domain of definition of f is $(-\delta, \delta)$ for a suitable $\delta > 0$.

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Now, we will calculate y', in particular we will derive the equation (2) by x taking into account that $y = y(x)$ is a composite function:

$$\begin{aligned} 2(x^2+y^2)\cdot(2x+2yy') - 6xy - 3x^2y' - 3y^2y' &= 0 \qquad (3) \\ y'\left(4y(x^2+y^2) - 3x^2 - 3y^2\right) &= 6xy - 4x(x^2+y^2) \\ y' &= \frac{6xy - 4x(x^2+y^2)}{(4y-3)(x^2+y^2)} \\ y'(0) &= 0 \end{aligned}$$

Check that the result is the same as if you would apply formula (1).

To obtain the second derivative, we derive the equation (3) once more by x again taking into account that $y = y(x)$ is a composite function. We obtain

$$2\left[(2x+2yy')^2 + (x^2+y^2)(2+2y(y')^2+2yy'')\right] - 6y - 12xy' - 3x^2y'' - 6y(y')^2 - 3y^2y'' = 0$$

$$y'' = \frac{-8(x+yy')^2 - 4(x^2+y^2)(1+y(y')^2) + 6y + 12xy' + 6y(y')^2}{(4y-3)(x^2+y^2)}$$

$$y''(0) = 2\,.$$

From the results we can see for example that the function $y = f(x)$ has a local minimum at the point $x = 0$, $f(0) = 1$.

Implicit function of n variables

Definition We say that the equation $F(x_1, x_2, \dots, x_n, z) = 0$ defines in the neighborhood of the point $(x_1^0, x_2^0, \dots, x_n^0, z_0)$ implicitly the function $z = f(x_1, \dots, x_n)$, if and only if

1. $F(x_1^0, x_2^0, \dots, x_n^0, z_0) = 0$,
2. $\exists \delta > 0,\ \epsilon > 0 \ : \ \forall x = (x_1, x_2, \dots, x_n) \in \mathcal{O}_\delta(x_1^0, x_2^0, \dots, x_n^0)$ is $z = f(x_1, x_2, \dots, x_n)$ the only number in the interval $(z_0 - \epsilon,\ z_0 + \epsilon)$, that satisfies the equation $F(x_1, x_2, \dots, x_n, z) = 0$.

Theorem The existence of the implicit function of n variables

Let $F \in \mathrm{C}^k(G)$, $G \subset \mathbb{R}^{n+1}$ is an open set, $k \geq 1$. Let $\mathbf{x} = (x_1, \ldots, x_n)$, $\mathbf{x}_0 = (x_1^0, x_2^0, \ldots, x_n^0)$ and let $(\mathbf{x}_0, z_0) \in G$ be such a point that

1. $F(\mathbf{x}_0, z_0) = 0$,
2. $\dfrac{\partial F}{\partial z}(\mathbf{x}_0, z_0) \neq 0$.

Then the equation $F(\mathbf{x}, z) = 0$ defines in the neighborhood of the point $(\mathbf{x}_0, z_0)$ implicitly a function $z = f(x_1, x_2, \ldots, x_n)$. Moreover, $f \in \mathrm{C}^k(\mathcal{O}_\delta(\mathbf{x}_0))$ for a certain $\delta > 0$.

The partial derivatives of the function $f(x_1, x_2, \ldots, x_n)$ are given by the formula

$$\frac{\partial f}{\partial x_i}(\mathbf{x}) = -\frac{\dfrac{\partial F}{\partial x_i}(\mathbf{x}, f(\mathbf{x}))}{\dfrac{\partial F}{\partial z}(\mathbf{x}, f(\mathbf{x}))}, \quad \text{for} \quad \mathbf{x} \in \mathcal{O}_\delta(\mathbf{x}_0), \quad i = 1, \ldots, n .$$

Remark The similar theorem is valid if we like to express for example x_1 as a function of $x_2, \ldots, x_n, z$, i.e. we want to have $x_1 = \psi(x_2, \ldots, x_n, z)$ in the neighborhood of the point $(\mathbf{x}_0, z_0)$ for which

$$F(\mathbf{x}_0, z_0) = 0 \quad and \quad \frac{\partial F}{\partial x_1}(\mathbf{x}_0, z_0) \neq 0 .$$

★ Example

Example By the application of the total differential, we compute the change of the volume of one mol of the gas governed by the van der Waals equation

$$\left(p+\frac{a}{V^2}\right)(V-b)=RT\,,\quad \text{where } a,\ b \text{ are constants},\ R \text{ is the gas constant}\,,$$

if the pressure p will change by $\mathrm{d}p$ and temperature T will change by $\mathrm{d}T$.

Solution $V=V(p,T)$, V is given implicitly. The approximate change of the volume ($\mathrm{d}V$ ... total differential $\doteq$ diference $V_{new}-V_{old}$) is

$$\mathrm{d}V=\frac{\partial V}{\partial p}\mathrm{d}p+\frac{\partial V}{\partial T}\mathrm{d}T\,.$$

Let us put

$$\underbrace{F(p,T,V)}_{\text{the function of three variables}}=(p+\frac{a}{V^2})(V-b)-RT$$

$$\underbrace{F(p,T,V)=0}_{\text{equation for the implicit function } V=V(p,T)}\ \ldots\ \text{the zero level set of } F$$

A simple example - a state equation

$$\frac{\partial F}{\partial p} = V - b\,, \quad \frac{\partial F}{\partial V} = -\frac{2a}{V^3}(V-b) + p + \frac{a}{V^2}\,, \quad \frac{\partial F}{\partial T} = -R$$

$$\frac{\partial V}{\partial p} = -\frac{\dfrac{\partial F}{\partial p}}{\dfrac{\partial F}{\partial V}} = -\frac{V-b}{-\dfrac{2a}{V^3}(V-b)+p+\dfrac{a}{V^2}} = \frac{(b-V)V^3}{pV^3 - aV + 2ab}$$

$$\frac{\partial V}{\partial T} = -\frac{\dfrac{\partial F}{\partial T}}{\dfrac{\partial F}{\partial V}} = -\frac{-RV^3}{-2a(V-b)+pV^3+aV} = \frac{RV^3}{pV^3 - aV + 2ab}$$

$$\mathrm{d}V = \frac{\partial V}{\partial p}\mathrm{d}p + \frac{\partial V}{\partial T}\mathrm{d}T = \frac{(b-V)V^3}{pV^3 - aV + 2ab}\mathrm{d}p + \frac{RV^3}{pV^3 - aV + 2ab}\mathrm{d}T$$

$$\mathrm{d}V = \frac{V^3}{pV^3 - aV + 2ab}\left((b-V)\mathrm{d}p + R\,\mathrm{d}T\right)$$

Remark $$(p+\frac{a}{V^2})(V-b)=RT \quad\Longrightarrow\quad F(p,T,V)=0\,.$$

V is given implicitly, p, T explicitly, we can express them directly from the equation:

$$p=\frac{RT}{V-b}-\frac{a}{V^2}\,,\qquad T=\frac{V-b}{R}(p+\frac{a}{V^2})\,.$$

Exercise

- ★ The same problem for the Soave–Redlich–Kwong state equation

$$p=\frac{RT}{V-b}-\frac{\alpha a}{V(V+b)}\,,\qquad \alpha\,,\ a\,,\ b\,,\ R \text{ are constants}\,.$$

- ★ The same problem for the Peng–Robinson state equation

$$p=\frac{RT}{V-b}-\frac{\alpha(T)}{V(V+b)+b(V-b)}\,,\qquad \alpha(T)=\left(1+k(1-\sqrt{\frac{T}{T_c}})\right)^2\,,$$

T_c ... the critical temperature, b, R, k constants.

- ★ Find out if in the neighborhood of the point $A=(1,1,1)$ is by the equation $3y^4-x^4z+4xyz^2-7yz^3+1=0$ implicitly defined a function $z=f(x,y)$. Is the point $(1,1)$ a stationary point of the function f?

★ Implicit functions – a general theorem

Let $(x, y) = ((x_1, \dots, x_m), (y_1, \dots, y_n)) \in \mathbb{R}^m \times \mathbb{R}^n$, let

$$F(x, y) = \underbrace{(F_1, \dots, F_n)}_{n \text{ equations}}\underbrace{(x, y)}_{m+n \text{ variables}}, \text{ i.e., the mapping } F : \mathbb{R}^{m+n} \longrightarrow \mathbb{R}^n,$$

$$F(x, y) = \begin{pmatrix} F_1(x_1, \dots, x_m,\ y_1, \dots, y_n) \\ F_2(x_1, \dots, x_m,\ y_1, \dots, y_n) \\ \vdots \\ F_n(x_1, \dots, x_m,\ y_1, \dots, y_n) \end{pmatrix} \in \mathbb{R}^n .$$

$D_x F \dots$ partial differential of F represented by a matrix $\left(\frac{\partial F_i}{\partial x_j}\right)_{\substack{i=1,\dots,n \\ j=1,\dots,m}}$ (Jacobi matrix $n \times m$)

Similarly:

$D_y F \dots$ partial differential of F represented by a matrix $\left(\frac{\partial F_i}{\partial y_j}\right)_{i,j=1,\dots,n}$ (Jacobi matrix $n \times n$)

$$D_xF = \begin{pmatrix} \frac{\partial F_1}{\partial x_1}, \frac{\partial F_1}{\partial x_2}, \dots, \frac{\partial F_1}{\partial x_m} \\ \frac{\partial F_2}{\partial x_1}, \frac{\partial F_2}{\partial x_2}, \dots, \frac{\partial F_2}{\partial x_m} \\ \vdots \\ \frac{\partial F_n}{\partial x_1}, \frac{\partial F_n}{\partial x_2}, \dots, \frac{\partial F_n}{\partial x_m} \end{pmatrix}, \quad D_yF = \begin{pmatrix} \frac{\partial F_1}{\partial y_1}, \frac{\partial F_1}{\partial y_2}, \dots, \frac{\partial F_1}{\partial y_n} \\ \frac{\partial F_2}{\partial y_1}, \frac{\partial F_2}{\partial y_2}, \dots, \frac{\partial F_2}{\partial y_n} \\ \vdots \\ \frac{\partial F_n}{\partial y_1}, \frac{\partial F_n}{\partial y_2}, \dots, \frac{\partial F_n}{\partial y_n} \end{pmatrix}.$$

Remark Notation:

$A \subset \mathbb{R}^{m+n}$ is an open set, $F : A \longrightarrow \mathbb{R}^n$, $F \in \mathrm{C}^r(A)$,

$F = (F_1, \dots, F_n)$, $\quad (x, y) = (x_1, \dots, x_m,\ y_1, \dots, y_n)$,

$F_i(x, y) = F_i(x_1, \dots, x_m,\ y_1, \dots, y_n)$, $\ i = 1, \dots, n$.

★ The general implicit function theorem

Let $(X_0, Y_0) = (x_1^0, x_2^0, \dots, x_m^0,\ y_1^0, y_2^0, \dots, y_n^0) \in A$ is such a point that

a) $F(X_0, Y_0) = 0$,

b) $\det(D_y F)\big|_{(X_0, Y_0)} \neq 0$.

Then there exists a neighborhood $\mathcal{B}$ of the point $X_0 \in \mathbb{R}^m$ and a uniquely defined mapping

$g : \mathcal{B} \longrightarrow \mathbb{R}^n$, $g = (g_1, \dots, g_n)$ such that

1.

$$\begin{aligned} y_1^0 &= g_1(x_1^0, \dots, x_m^0) \\ y_2^0 &= g_2(x_1^0, \dots, x_m^0) \\ &\vdots \\ y_n^0 &= g_n(x_1^0, \dots, x_m^0) \end{aligned}$$

2.

$$F(\underbrace{x, g(x)}_{(x_1, x_2, \dots, x_m, g_1(x_1, \dots, x_m), \dots, g_n(x_1, \dots, x_m))}) = 0 \quad \forall x \in \mathcal{B}$$

Moreover, $g \in C^r(\mathcal{B})$, $D(g(x)) = -\left(D_y F(x, g(x))\right)^{-1} \left(D_x F(x, g(x))\right)$.

Remarks

ad a) If we rewrite the equation $F(X_0, Y_0) = 0$ in components, we obtain n equations

$$\begin{aligned} F_1(x_1^0, x_2^0, \ldots, x_m^0, y_1^0, y_2^0, \ldots, y_n^0) &= 0 \\ F_2(x_1^0, x_2^0, \ldots, x_m^0, y_1^0, y_2^0, \ldots, y_n^0) &= 0 \\ &\vdots \\ F_n(x_1^0, x_2^0, \ldots, x_m^0, y_1^0, y_2^0, \ldots, y_n^0) &= 0 \end{aligned}$$

ad b) The assumption b) says, that the partial differential of F by y in the point (X_0, Y_0) is a regular $n \times n$ matrix, in other words $\operatorname{rank}(D_y F \big|_{(X_0, Y_0)}) = n$. The last equality is a matrix equation. Let us check the dimensions.

$$\underset{n \times m}{D(g(x))} = -\underset{n \times n}{(D_y F(x, g(x)))^{-1}} \underset{n \times m}{(D_x F(x, g(x)))} .$$

★ Derivative of the composed functions

Remark Let $m = 1,\ n = 2$

$$F_i(x, y_1, y_2) = 0\,, \quad y_1 = g_1(x), \quad y_2 = g_2(x)\,, \quad i = 1, 2\,.$$

Let us derive the equation by x:

$$\begin{array}{lrcl} i = 1 & \dfrac{\partial F_1}{\partial x}\cdot 1 + \dfrac{\partial F_1}{\partial y_1}\cdot g_1'(x) + \dfrac{\partial F_1}{\partial y_2}\cdot g_2'(x) & = & 0\,,\\ i = 2 & \dfrac{\partial F_2}{\partial x}\cdot 1 + \dfrac{\partial F_2}{\partial y_1}\cdot g_1'(x) + \dfrac{\partial F_2}{\partial y_2}\cdot g_2'(x) & = & 0\,,\end{array}$$

$$\Longrightarrow \underbrace{\begin{pmatrix} \dfrac{\partial F_1}{\partial y_1} & \dfrac{\partial F_1}{\partial y_2} \\ \dfrac{\partial F_2}{\partial y_1} & \dfrac{\partial F_2}{\partial y_2} \end{pmatrix}}_{D_g F,\ 2\times 2} \cdot \underbrace{\begin{pmatrix} g_1'(x) \\ g_2'(x) \end{pmatrix}}_{D\dot{g},\ 2\times 1} = - \underbrace{\begin{pmatrix} \dfrac{\partial F_1}{\partial x} \\ \dfrac{\partial F_2}{\partial x} \end{pmatrix}}_{D_x F,\ 2\times 1}$$

$D_g F\,\big|_{(X_0, Y_0)}$ is a regular matrix $\Longrightarrow$ $\exists$ an inverse matrix. Let us recall that $g : \mathcal{B} \subset \mathbb{R} \longrightarrow \mathbb{R}^2\,,\ (x_0, y_0) \in \mathcal{B}\,.$

★

We obtain

$$\begin{pmatrix} g_1'(x) \\ g_2'(x) \end{pmatrix} = -\begin{pmatrix} \frac{\partial F_1}{\partial y_1} & \frac{\partial F_1}{\partial y_2} \\ \frac{\partial F_2}{\partial y_1} & \frac{\partial F_2}{\partial y_2} \end{pmatrix}^{-1} \begin{pmatrix} \frac{\partial F_1}{\partial x} \\ \frac{\partial F_2}{\partial x} \end{pmatrix}$$

$$Dg(x) = -(D_y F(x, g(x)))^{-1}(D_x F(x, g(x)))$$

Example Let $F(x, y) = 0$, where $F = (F_1, F_2)$, $F : \mathbb{R}^4 \longrightarrow \mathbb{R}^2$, $F_1,\ F_2 \in \mathcal{C}^\infty(\mathbb{R}^4)$, $(x, y) = (x_1, x_2, y_1, y_2)$, t.j. $m = 2,\ n = 2$,

$$\begin{aligned} F_1(x, y) &= x_1^2 + 2x_2 + y_1^2 + 2y_2 - 8 = 0, \quad & F_1 : \mathbb{R}^4 \longrightarrow \mathbb{R}^1, \\ F_2(x, y) &= x_1 - x_2^2 + y_1 - y_2^2 + 3 = 0 \quad & F_2 : \mathbb{R}^4 \longrightarrow \mathbb{R}^1 \end{aligned}$$

Let $X_0 = (1, 1)$, $Y_0 = (1, 2)$.

★ Solution of the example

At first, we have to check the assumptions of the existence of the implicitly defined function $g(x)$ defined implicitly by the equation $F(x, y) = 0$ in the neighborhood $\mathcal{B}$ of the point $(1, 1, 1, 2)$:

$$\begin{aligned} F_1(1,1,1,2) &= 1+2+1+4-8=0 \\ F_2(1,1,1,2) &= 1-1+1-4+3=0 \end{aligned} \implies F(X_0, Y_0) = 0 \in \mathbb{R}^2$$

$$D_y F = \begin{pmatrix} \dfrac{\partial F_1}{\partial y_1} & \dfrac{\partial F_1}{\partial y_2} \\ \dfrac{\partial F_2}{\partial y_1} & \dfrac{\partial F_2}{\partial y_2} \end{pmatrix} = \begin{pmatrix} 2y_1 & 2 \\ 1 & -2y_2 \end{pmatrix}, \quad (D_y F)\big|_{(1,1,1,2)} = \begin{pmatrix} 2 & 2 \\ 1 & -4 \end{pmatrix}$$

$\det(D_y F)\big|_{(1,1,1,2)} = -10 \neq 0 \implies D_y F\big|_{(x_0,y_0)}$ is the regular matrix, its rank is 2 .

$\implies$ $\exists$ a neighborhood $\mathcal{B}$ of the point $(1, 1) \in \mathbb{R}^2$ and a uniquely defined function $g : \mathcal{B} \longrightarrow \mathbb{R}^2$ such that

$$g = g(x_1, x_2) = \begin{pmatrix} g_1(x_1, x_2) \\ g_2(x_1, x_2) \end{pmatrix}; \quad g(1,1) = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \begin{matrix} \longleftarrow y_1 \\ \longleftarrow y_2 \end{matrix} .$$

So, we proved that $g(X_0) = Y_0$, $g \in C^\infty(\mathbb{R}^2)$ and $F(x, g(x)) = 0 \ \forall\ x \in \mathcal{B}$.

Derivative:

$$Dg(x) = -\left(D_yF(x,g(x))\right)^{-1} \cdot \left(D_xF(x,g(x))\right), \quad D_xF = \begin{pmatrix} \frac{\partial F_1}{\partial x_1} & \frac{\partial F_1}{\partial x_2} \\ \frac{\partial F_2}{\partial x_1} & \frac{\partial F_2}{\partial x_2} \end{pmatrix}$$

$$Dg(x) = -\begin{pmatrix} 2y_1 & 2 \\ 1 & -2y_2 \end{pmatrix}^{-1} \cdot \begin{pmatrix} 2x_1 & 2 \\ 1 & -2x_2 \end{pmatrix}, \quad D_xF = \begin{pmatrix} 2x_1 & 2 \\ 1 & -2x_2 \end{pmatrix}$$

$$\begin{pmatrix} 2y_1 & 2 \\ 1 & -2y_2 \end{pmatrix}^{-1} = \frac{1}{4y_1y_2+2}\begin{pmatrix} 2y_2 & 2 \\ 1 & -2y_1 \end{pmatrix}$$

$$D(g(x)) = -\frac{1}{4g_1(x_1,x_2)g_2(x_1,x_2)+2}\begin{pmatrix} 4x_1g_2(x_1,x_2)+2 & 4g_2(x_1,x_2) \\ 2x_1 - 2g_1(x_1,x_2) & 2+4x_2g_1(x_1,x_2) \end{pmatrix}$$

$$Dg(x)|_{(1,1)} = -\frac{1}{5}\begin{pmatrix} 3 & 4 \\ 0 & 3 \end{pmatrix}$$

★ Example: Maximal profit

Let the production of a given firm is governed by the equation $y = 14x_1 + 11x_2 - x_1^2 - x_2^2$ and the profit is defined by the equation $\pi = py - w_1x_1 - w_2x_2$, where p is the given prize of the product, w_1 is salary of the first worker, w_2 is salary of the second worker.
The aim: to maximize the profit, i.e., we are seeking for $x_1 = g_1(p, w_1, w_2)$ and $x_2 = g_2(p, w_1, w_2)$ – such enters to the production that the profit π will be maximal.

We put production into the function for the profit

$$\pi(x_1, x_2) = 14px_1 + 11px_2 - px_1^2 - px_2^2 - w_1x_1 - w_2x_2$$

and we are looking for a maximum of this function:

$$\frac{\partial \pi}{\partial x_1} = 14p - 2px_1 - w_1 = 0 \quad \wedge \quad \frac{\partial \pi}{\partial x_2} = 11p - 2px_2 - w_2 = 0\,.$$

Let us set

$$\varphi_1(x_1, x_2, p, w_1, w_2) := 14p - 2px_1 - w_1\,, \quad \varphi_2(x_1, x_2, p, w_1, w_2) := 11p - 2px_2 - w_2\,.$$

★

We obtain the system of two implicit equations ($n = 2, m = 3$):

$$\begin{aligned} \varphi_1(x_1, x_2, p, w_1, w_2) &= 14p - 2px_1 - w_1 \,, \\ \varphi_2(x_1, x_2, p, w_1, w_2) &= 11p - 2px_2 - w_2 \,. \end{aligned}$$

The determinant of the Jacobi matrix of this system:

$$J = \begin{vmatrix} \frac{\partial \varphi_1}{\partial x_1} & \frac{\partial \varphi_1}{\partial x_2} \\ \frac{\partial \varphi_2}{\partial x_1} & \frac{\partial \varphi_2}{\partial x_2} \end{vmatrix} = \begin{vmatrix} -2p & 0 \\ 0 & -2p \end{vmatrix} = 4p^2 > 0 \,.$$

$J \neq 0 \quad \Longrightarrow$ we can apply the implicit function theorem.

Derivative:

$$\begin{aligned} \frac{\partial \varphi_1}{\partial x_1} \cdot \frac{\partial x_1}{\partial p} + \frac{\partial \varphi_1}{\partial x_2} \cdot \frac{\partial x_2}{\partial p} &= -\frac{\partial \varphi_1}{\partial p} \\ \frac{\partial \varphi_2}{\partial x_1} \cdot \frac{\partial x_1}{\partial p} + \frac{\partial \varphi_2}{\partial x_2} \cdot \frac{\partial x_2}{\partial p} &= -\frac{\partial \varphi_2}{\partial p} \end{aligned}$$

In the matrix form:

$$\begin{pmatrix} \frac{\partial \varphi_1}{\partial x_1} & \frac{\partial \varphi_1}{\partial x_2} \\ \frac{\partial \varphi_2}{\partial x_1} & \frac{\partial \varphi_2}{\partial x_2} \end{pmatrix} \begin{pmatrix} \frac{\partial x_1}{\partial p} \\ \frac{\partial x_2}{\partial p} \end{pmatrix} = - \begin{pmatrix} \frac{\partial \varphi_1}{\partial p} \\ \frac{\partial \varphi_2}{\partial p} \end{pmatrix}$$

$$\begin{pmatrix} \frac{\partial x_1}{\partial p} \\ \frac{\partial x_2}{\partial p} \end{pmatrix} = - \begin{pmatrix} -2p & 0 \\ 0 & -2p \end{pmatrix}^{-1} \cdot \begin{pmatrix} 14 - 2x_1 \\ 11 - 2x_2 \end{pmatrix} = \begin{pmatrix} \frac{7 - x_1}{p} \\ \frac{11 - 2x_2}{2p} \end{pmatrix} .$$

From the condition for extreme, we have:

$$14p - 2px_1 - w_1 = 0 , \qquad 11p - 2px_2 - w_2 = 0 \quad \Longrightarrow$$

$$x_1 = 7 - \frac{w_1}{2p} , \quad x_2 = \frac{11p - w_2}{2p} = \frac{11}{2} - \frac{w_2}{2p} ,$$

where p, w_1, w_2 are parameters of the system.

Recommended literature

- Bubeník F.: Mathematics for Engineers, textbook of Czech Technical University, Prague 2007
- Dontchev, A. L., Rockafellar, R. T.: Implicit Functions and Solution Mappings: A View from Variational Analysis (Springer Monographs in Mathematics). Springer, 2010
- Kaplan W.: Advanced Calculus. Boston: Addison-Wesley, 2003. ISBN 0-201-79937-5.
- Krantz, S.G., Parks, H. R.: The Implicit Function Theorem. History, Theory, and Applications. Modern Birkhäuser Classics. Springer New York, 2013.
- Loomis L. H., Sternberg S.: Advanced calculus, Jones and Bartlett Publishers, Boston, London, 1990, ISBN 0-86720-122-3, http://www.math.harvard.edu/ shlomo/docs/AdvancedCalculus.pdf
- Mellor W. J.: Higher Mathematics for Students of Chemistry and Physics, Cosimo, Inc., 2007. ISBN: 978-1-60206-569-7.
- Spiegel M., Wrede R.: Schaum's Outline of Advanced Calculus, Third Edition, The McGraw-Hill Companies, 2010, ISBN 978-0-07-162366-7 (eBook ISBN 978-0-07-162367-4).
- Steiner E.: The Chemistry Maths Book, 2nd Edition, Oxford University Press, 2008. ISBN 978-0-19-920535-6.