Mathematics for chemical engineers

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7. Qualitative theory of systems of ODEs

Introduction

Winter Semester 2016-2017

Outline

★ A universe in a jar

Example

A jar is filled with a nutritive solution and some bacteria. As time progresses, the bacteria reproduce (by dividing) and die. Let us denote
b (= birth) . . . the rate at which the microbes reproduce,
p (= perish) . . . the rate at which they die.
The population is growing at the rate $r = b - p$. This means that if there are x bacteria in the jar, then the rate at which the number of bacteria is increasing is $(b - p)x$. We obtain

$$\frac{\mathrm{d}x}{\mathrm{d}t} = rx.$$

If we begin with x_0 bacteria at time $t = 0$, then

$$x(t) = \mathrm{e}^{rt} x_0 \,, \quad x(0) = x_0 \,.$$

In the short run, this makes sense. The formula says that there are x_0 bacteria at time $t = 0$ and then the number grows at an exponential rate. However, as time goes on, the number of bacteria will be exceedingly large (larger than the number of atoms in the universe). Thus this simple model is not realistic in the long time period.

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As the number of bacteria reproduce, they tend to crowd each other, produce toxic waste products, etc. It makes sense to postulate that a death rate increases with the population. Instead of a constant death rate, let us suppose that the death rate is px. So if there are x bacteria, they are decreasing in number at a rate px^2. We obtain the dynamical system

$$\frac{\mathrm{d}x}{\mathrm{d}t} = bx - px^2. \tag{1}$$

Question: Is there a self-sustaining population in this model ?
So, we are looking for a number $\tilde{x}$ for which $b\tilde{x} - p\tilde{x}^2 = 0$. At this special level, the reproduction and death rates are exactly in balance, the population is neither increasing nor decreasing. By setting the right-hand-side of equation (1) equal to zero we get

$$\frac{\mathrm{d}x}{\mathrm{d}t} = bx - px^2 = 0 \;\Leftrightarrow\; x = 0 \,\vee\, b - px = 0 \;\Leftrightarrow\; x = 0 \vee x = \frac{b}{p}.$$

There are two self-sustaining population levels (equilibria): $x = 0$ and $x = \dfrac{b}{p}$.

These two values correspond to the two roots of the quadratic equation $bx - px^2 = 0$. This is the equation of a parabola.

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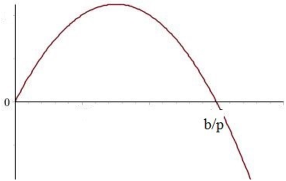

Let's consider $\tilde{x} = 0$. Clearly this is self-sustaining: There are no bacteria, so none can be born and none can die. Forever there will be no bacteria in the jar. But with a slightest contamination $0 < x < \frac{b}{p}$ we see that

$$\frac{\mathrm{d}x}{\mathrm{d}t} = bx - px^2 > 0 \,.$$

Thus the number of bacteria will start to increase as soon as the jar has been contaminated.

The equilibrium $\tilde{x} = 0$ is unstable. Slight perturbations away from this equilibrium will destroy the equilibrium.

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Now, let us consider $\tilde{x} = \frac{b}{p}$. At this population level, birth and death rates are exactly in balance:

$$b\tilde{x} = b \cdot \frac{b}{p} = \frac{b^2}{p}, \quad p\tilde{x}^2 = p\left(\frac{b}{p}\right)^2 = \frac{b^2}{p}.$$

Let us consider what happens in the case the population x is slightly above or slightly below $\tilde{x} = \frac{b}{p}$.

- If $x > \tilde{x}$ then $\frac{\mathrm{d}x}{\mathrm{d}t} < 0 \Rightarrow$ the number of bacteria will drop back toward $\frac{b}{p}$.
- Conversely, if $x < \tilde{x}$ then $\frac{\mathrm{d}x}{\mathrm{d}t} > 0$ and the population will tend to increase back toward $\frac{b}{p}$.

Small perturbations away from $\tilde{x} = \frac{b}{p}$ will self-correct back to $\frac{b}{p}$.

The equilibrium $\tilde{x} = \frac{b}{p}$ is stable.

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Analytic solution (Matlab, Mathematica, Maple)

$$x(t) = \frac{x_0 b e^{bt}}{(b - p x_0) + p x_0 e^{bt}} .$$

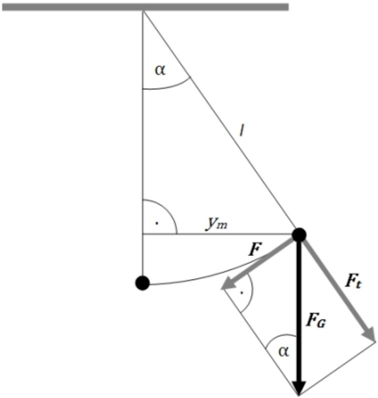
If $x_0 > 0$, then

$$\lim_{t \to \infty} \frac{x_0 b e^{bt}}{(b - p x_0) + p x_0 e^{bt}} = \frac{b}{p} ,$$

hence for $t \to \infty$ the system will "finish"in the state $\tilde{x} = \dfrac{b}{p}$.

★ Example: Mathematical pendulum

Example Let mathematical pendulum in the constant gravitation field has a small weight with mass m hanging on the thread of length l of negligible mass. The angle α should not be greater than 5°.
The phase space M is the set of all possible positions of the pendulum, that are represented by an angle and by an angular velocity $\implies$ M is a two dimensional cylinder and dynamics of the system is smooth movement in M.



We decompose the gravitational force F_G into the direction of the prolongation of the thread and into the direction perpendicular to it. The F_t is canceled by the strength of the fiber and does not affect the movement. The reason why the pendulum is moving is the force F. We express $\sin\alpha$ from both right-angled triangles .

$$\sin\alpha = \frac{F}{F_G}, \qquad \sin\alpha = \frac{y_m}{l} \quad \Longrightarrow \quad F = \frac{F_G y_m}{l} = \frac{mgy_m}{l}\,.$$

The force law $\quad F = ma_m \quad \Longrightarrow \quad ma_m = \dfrac{mgy_m}{l} \quad \Longrightarrow \quad a_m = \dfrac{gy_m}{l}\,.$

For instant acceleration value of oscillatory motion, the following equation holds

$$a = a_m \sin\omega t = \omega^2 y_m \sin\omega t\,,$$

i.e., after some manipulation and substitution for angular frequency we obtain

$$\omega = \sqrt{\frac{g}{l}}, \quad \Longrightarrow \quad T = 2\pi\sqrt{\frac{l}{g}}\,.$$

The period of oscillation of the mathematical pendulum does not depend on the weight m or on the size of the deviation y_m. Due to the constant value of gravity the acceleration period depends only on the length of the pendulum.

Dynamical system

Dynamical system . . . the system that evolves in time denoted as t.
Phase space of the system . . . the set of all possible states, i.e., values of variables. We will consider only systems with finite dimensional phase space, i.e., the state of the system will be described by values of the finite number of variables.
Notation: x . . . state of the system, M . . . phase space
The state of the system is fully described by variables $x \in M$ and also by some values of parameters. Behavior of the dynamical system is usually modeled as behavior of the system of differential equations.

Differential equations describe the relations between the function and its derivatives (1 variable – ODE, more variables – PDE).
System of ODE

$$\mathbf{F}\left(t, \mathbf{y}, \frac{\mathrm{d}\mathbf{y}}{\mathrm{d}t}, \ldots \frac{\mathrm{d}^k\mathbf{y}}{\mathrm{d}t^k}\right) = \mathbf{0}\,,$$

$$\mathbf{y} : \mathbb{R} \longrightarrow N \subseteq \mathbb{R}^d\,, \quad \text{notation: } \frac{\mathrm{d}\mathbf{y}}{\mathrm{d}t} := \dot{\mathbf{y}}\,,$$

the graph of a particular solution . . . a trajectory

$$\mathcal{C} = \{\mathbf{y}(t),\ t \in \mathbb{R}\} \ \ldots \text{ a curve in } N \subseteq \mathbb{R}^d$$

Any explicit differential equation can be rewritten as a system of ordinary differential equations of the 1st order in the following way: We introduce new variables

$$\frac{\mathrm{d}^k \mathbf{y}}{\mathrm{d}t^k} = \mathbf{G}\left(t, \mathbf{y}, \dot{\mathbf{y}}, \dots, \frac{\mathrm{d}^{k-1}\mathbf{y}}{\mathrm{d}t^{k-1}}\right)$$

$$\begin{aligned} \mathbf{x}_1 &:= \mathbf{y} \\ \mathbf{x}_2 &:= \dot{\mathbf{y}} = \frac{\mathrm{d}\mathbf{y}}{\mathrm{d}t}, \ \dots\ \mathbf{x}_i := \frac{\mathrm{d}^{i-1}\mathbf{y}}{\mathrm{d}t^{i-1}} \ \dots\ \mathbf{x}_k = \frac{\mathrm{d}^{k-1}\mathbf{y}}{\mathrm{d}t^{k-1}} \end{aligned}$$

and we obtain the system

$$\begin{aligned} \frac{\mathrm{d}\mathbf{x}_i}{\mathrm{d}t} &= \mathbf{x}_{i+1}, \ i = 1, 2, \dots, k-1, \qquad \mathbf{x}_i = \underbrace{(x_1^{(i)}, x_2^{(i)}, \dots, x_d^{(i)})}_{d \text{ variables}} \\ \frac{\mathrm{d}\mathbf{x}_k}{\mathrm{d}t} &= \mathbf{G}(t, \mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k), \qquad k \text{ equations} \end{aligned}$$

This is a system of $n = k \cdot d$ first order differential equations in the phase space M, $\dim M = k \cdot d = n$.

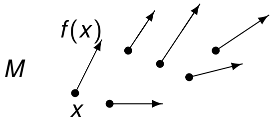
In what follows, we will consider $x : \mathbb{R} \longrightarrow M$, $f(t, x)$ represents velocity in time t and point x.

$$\Longrightarrow \qquad f : \mathbb{R} \times M \longrightarrow \mathbb{R}^n .$$

A particular case: autonomous differential equation – the right-hand side f is explicitly not a function of t

$$\Longrightarrow \qquad \dot{x} = f(x) , \quad f : M \longrightarrow \mathbb{R}^n$$

f . . . velocity in each point of the phase space M



In general, with a few exceptions, there is no reason to deal with non-autonomous systems because every non-autonomous system can be rewritten as an autonomous, if we introduce a new variable, eg. $x_{n+1} = t$. Then for this variable $\dot{x}_{n+1} = 1$.

★ Example

$$y'' = -y, \quad y(0) = 0, \quad y'(0) = 1 \,. \tag{2}$$

Let us set $y_1(x) := y(x)\,, \quad y_2(x) := y'(x)\,.$ Then

$$\begin{aligned} y_1' &= y'(x) = y_2\,, \\ y_2' &= y'' = -y = -y_1\,. \end{aligned} \tag{3}$$

We obtain a linear system of two differential equations with initial conditions

$$\begin{aligned} y_1' &= y_2\,, & y_1(0) &= 0\,, \\ y_2' &= -y_1\,, & y_2(0) &= 1\,. \end{aligned} \tag{4}$$

Exercise: Solve the system (3), (4), and compute the particular solution of the equation (2) with initial conditions (4). Check that both solutions are correct (the same?).

Remark:

Initial (Cauchy) problem – first order differential equaation

$$\dot{x} = f(x)\,, \quad x(t_0) = x_0\,.$$

We are looking for the solution $x(t)$, that fulfills the initial condition in the given time t_0.

Initial (Cauchy) problem of the second order

$$\ddot{x} = f(x)\,, \quad x(t_0) = x_0\,, \quad x'(t_0) = x_1\,.$$

We are seeking for the soloution $x(t)$, that fulfills the initial conditions in the given time t_0.

Boundary problem of the second order

$$\ddot{x} = f(x)\,, \quad x(t_0) = x_0\,, \quad x(t_1) = x_1\,.$$

We are looking for the solution $x(t)$, that satisfies the given boundary conditions .

General solution $x(t; c)$ is such a solution that depends only on general parameters c , i.e. no initial or boundary conditions are given.

★ Logistic equation

One dimensional autonomous initial problem (can be solved e.g. by separation of the variables):

$$\dot{x} = f(x)\,, \qquad x(0) = x_0\,.$$

Example One of the simplest nonlinear ordinary differential equations – logistic equation

$$\underbrace{\dot{x} = r\,x(1-x)}_{\text{the simplest model of the growing of population}}\,, \quad x(0) = x_0\,.$$

$x = \dfrac{N}{K}$, $N(t)$ ... number of individuals in a population over time t, K ... carrying capacity, $r = b - p$ ... the difference between the coefficients of birth and death of the population, $r << K$.

Solution by separation

$$\dot{x} = r\,x(1-x) = 0 \iff x = 0 \ \vee \ x = 1 \quad \Longrightarrow$$

two constant solutions: $x(t) = 0\ \forall\, t \in \mathbb{R}$ and $x(t) = 1\ \forall\, t \in \mathbb{R}$.

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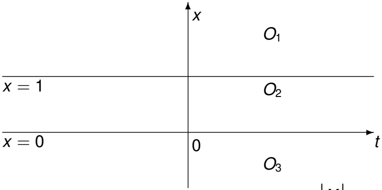
Our equation has the form

$$\dot{x} = f(t,x) = g(t)\cdot h(x)\,,\ \text{here } g(t) = r,\ h(x) = x(x-1)\,.$$

The constant solutions divide plane $x - y$ into three subregions O_1, O_2, O_3.



$$\begin{aligned}
\frac{\mathrm{d}x}{\mathrm{d}t} &= r\,x(x-1)\,,\\
\frac{\mathrm{d}x}{x(x-1)} &= r\mathrm{d}t\\
\int \frac{1}{x}\mathrm{d}x + \int \frac{1}{1-x}\mathrm{d}x &= r\int \mathrm{d}t\\
\ln \frac{|x|}{|1-x|} &= r\,t + c,\ c \in \mathbb{R}
\end{aligned}$$

$$\text{in } O_1\colon x > 1 \implies \frac{|x|}{|1-x|} = \frac{x}{x-1}$$

$$\text{in } O_2\colon 0 < x < 1 \implies \frac{|x|}{|1-x|} = \frac{x}{1-x}$$

$$\text{in } O_3\colon x < 0 \implies \frac{|x|}{|1-x|} = -\frac{x}{1-x} = \frac{x}{x-1}$$

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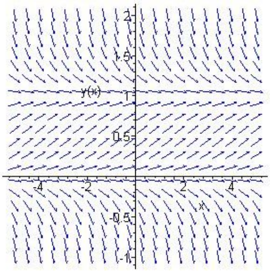
$$\ln\left|\frac{x}{1-x}\right| = r\,t + c\,,\quad x(0) = x_0 \quad\Longrightarrow\quad \ln\left|\frac{x_0}{1-x_0}\right| = c$$

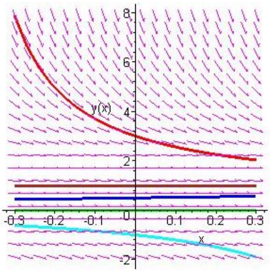
$$\ln\left|\frac{x}{1-x}\right| - \ln\left|\frac{x_0}{1-x_0}\right| = r\,t$$

$$x(t) = \frac{x_0}{x_0 + (1-x_0)\mathrm{e}^{-r\,t}} \quad \text{. . . a general solution of the logistic equation}$$

$$+\ \text{constant solutions}\qquad \left.\begin{array}{lll} x(t) = 0 & \forall t \in \mathbb{R} & (\text{pro } x_0 = 0) \\ x(t) = 1 & \forall t \in \mathbb{R} & (\text{pro } x_0 = 1) \end{array}\right\} \text{steady states}$$



Vector field, $r = 0.9$



Phase portrait, $r = 0.9$

★ Another example

It is not always possible to express the general solution explicitly.

$$\dot{x} = f(x)\,, \qquad f(x) = -\frac{x}{1+x^2}\,, \quad x(0) = x_0\,.$$

$$\underbrace{f(x) = 0 \iff x = 0}_{\text{equilibrium } x=0} \quad \Longrightarrow \quad \text{a stationary solution } x(t) = 0,\ t \in \mathbb{R}$$

Let $x \neq 0$. Then we solve the equation by separation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -\frac{x}{1+x^2} \quad \Longrightarrow \quad \int \frac{1+x^2}{x}\mathrm{d}x = -\int \mathrm{d}t$$

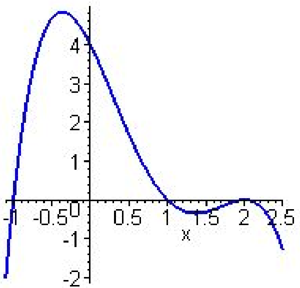
$$\ln|x| + \frac{x^2}{2} = -t + C\,, \qquad x(0) = x_0\,, \quad \text{i.e.,} \quad \ln|x_0| + \frac{x_0^2}{2} = C$$

The general solution:

$$\ln|x| + \frac{x^2}{2} = -t + \ln|x_0| + \frac{x_0^2}{2} \tag{5}$$

From the equation (5) it is not possible to express explicitly $x(t)$.

Back to dynamics



graph$(f) = \{(x, y), x \in M, y = f(x)\}$

3 equilibria

$$\dot{x} = f(x)\,, \quad f(x)\ldots \text{velocity in } x \in M\,,$$

$M \ldots$ phase space.

In one dimension, there are only three possibilities:

$$f(x) > 0 \ldots \text{ positive velocity}$$

$$f(x) < 0 \ldots \text{ negative velocity}$$

$$f(x) = 0 \ldots \text{ equilibrium}$$

What we can see from the graph of f?

If $f(x_0) > 0$, the "movement"is to the right and $x(t)$ grows monotonically until $f(x(t)) > 0$. If x^* is the first zero of f "after"x_0 and $f(x) > 0$ in $\langle x_0, x^*)$, then $x(t) \longrightarrow x^*$ for $t \longrightarrow \infty$. If there is no zero of f after x_0, then $x(t) \rightarrow \infty$ for $t \rightarrow \infty$. Similarly, if $f(x_0) < 0$, the "movement"goes to the left and $x(t)$ is decreasing monotonically ...

Summary: The dynamics of one dimensional autonomous ordinary differential equation is simple: the trajectory tends monotonically to the equilibrium or to the infinity.

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Exercise
In the logistic equation put $r = 1$, i.e.,

$$\dot{x} = x(1 - x), \quad x(0) = x_0 \, .$$

Draw an appropriate parabola and illustrate in this figure that

$x(t)$ is increasing on $(0, 1)$ and

$x(t)$ is decreasing on $(-\infty, 0)$ and on $(1, +\infty)$.

★ Two-dimensional population dynamics

Let us consider the model behavior of the two populations. One population – predators – feed on other population – prey, that has a different food. The model is a variation of the Lotka–Volterra system (a special case of Bazykin's ecological model).

Bazykin's model

$$\dot{x} = x - \frac{xy}{1+\alpha x} - \varepsilon x^2 \,, \tag{6}$$

$$\dot{y} = -\gamma y + \frac{xy}{1+\alpha x} - \delta y^2 \,, \tag{7}$$

where $x \ldots$ population size of prey, $y \ldots$ predator population size, $\alpha \geq 0 \ldots$ constant of saturation of predators, $\gamma \geq 0 \ldots$ constant of mortality of predators, nonnegative constants $\varepsilon, \delta \ldots$ constants of rivalry between prey and predators.

We add initial conditions

$$x(0) = x_0 \,, \qquad y(0) = y_0 \,.$$

★ Predator and prey

So far we considered a simple model of a biological system involving only one species. Now we consider a more complex model involving two species. Let the first species (the prey) be, say, rabbits whose population at time t is $r(t)$. The second species (the predator), say wolves, feeds on the prey. The population of wolves is $w(t)$ at time t.

Left on their own the rabbits will reproduce with rate $\frac{\mathrm{d}r}{\mathrm{d}t} = ar$ for some positive constant a. The wolves, on the other hand, will starve without rabbits to eat and their population will decline: $\frac{\mathrm{d}w}{\mathrm{d}t} = -bw$ for some $b > 0$.

However, when brought into the same environment, the wolves will eat the rabbits with the expected effects on each population: more wolves, fewer rabbits. Suppose there are w wolves and r rabbits. What is the likelihood that a wolf will catch a rabbit? The more wolves or the more rabbits there are, the more likely that a wolf will meet a rabbit. For this reason, we assume there is loss of the rabbit population proportional to rw and a gain of the wolf population, also proportional to rw.

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We write these changes in the population as follows:

$$\begin{aligned} \frac{\mathrm{d}r}{\mathrm{d}t} &= ar - grw \\ \frac{\mathrm{d}w}{\mathrm{d}t} &= -bw + hrw \,, \end{aligned}$$

where a, b, g, h are positive constants.
We can write this pair of differential equations in the form $\dot{x} = f(x)$, i.e.,

$$\begin{bmatrix} \dot{r} \\ \dot{w} \end{bmatrix} = \begin{bmatrix} ar - grw \\ -bw + hrw \end{bmatrix} .$$

We can approximate the solution of this system of differential equations numerically.

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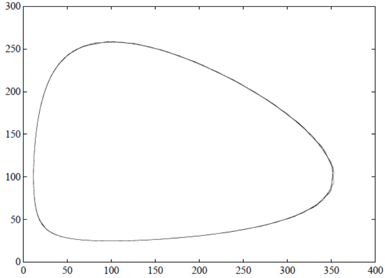

Phase portrait for predator-prey model. Horizontal axis is the number of prey (rabbits), and the vertical axis is the number of predators (wolves).

The population behavior is periodic. When there are few wolves, the rabbit population soars. Then, as food (i.e., rabbits) becomes more plentiful, the wolf population rises. But as the wolf population climbs, the wolves overhunt the rabbits, and the rabbit population falls. This causes food to become scarce for the wolves, and their numbers fall in turn. Finally, the number of wolves is low enough for the rabbit population to begin to recover, and the cycle begins again.

Two-dimensional dynamics

$$z = \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 : \quad \dot{z} = f(z) = \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} P(x,y) \\ Q(x,y) \end{pmatrix} . \tag{8}$$

At first, we are looking for a set S of equilibrium states,

$$S = \{(x,y) \in \mathbb{R}^2 : \quad P(x,y) = Q(x,y) = 0\} .$$

More information about system behavior is obtained by using the so-called nullclins. A nullclin is a curve, on which one of the velocity components of $f(z)$ is zero., i.e.,

$$\begin{aligned} N_x &= \{(x,y) : P(x,y) = 0\} \implies \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 \\ Q(x,y) \end{pmatrix} , \\ N_y &= \{(x,y) : Q(x,y) = 0\} \implies \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} P(x,y) \\ 0 \end{pmatrix} . \end{aligned}$$

$$\begin{aligned} \text{equilibrium} &= \text{intersection of nullclins} \\ \mathbf{S} &= \mathbf{N_x} \cap \mathbf{N_y} . \end{aligned}$$

★ Lotka–Volterra dynamical system

Example Lotka-Volterra dynamical system of two competing species is described by a system

$$\dot{x} = x(a - bx - cy)\,, \tag{9}$$
$$\dot{y} = y(d - ex - fy)\,, \tag{10}$$

where x, y are populations $\Rightarrow$ $x \geq 0, y \geq 0$, contants a, b, c, d, e, f are in biological applications positive, i.e., phase space

$$M = \{(x, y) \in \mathbb{R}^2 : x \geq 0, y \geq 0\}\,.$$

Let us compute the nullclin N_x:
$x(a - bx - cy) = 0 \Leftrightarrow x = 0 \vee a - bx - cy = 0 \Rightarrow$

$$N_x = \{(x, y) \in M, \underbrace{x = 0}_{\text{axis } y}\} \cup \{(x, y) \in M, y = \frac{a - bx}{c}\}\,.$$

Similarly for the nullclin N_y we have

$$N_y = \{(x, y) \in M, \underbrace{y = 0}_{\text{axis } x}\} \cup \{(x, y) \in M, y = \frac{d - ex}{f}\}\,.$$

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The set of equilibrium states $S = N_x \cap N_y$ usually consists from the points exceptionally it may include a straight line.

Remark The steady state corresponds to intersection of one curve in N_x with a curve in N_y. For example the intersection of $x = 0$ with $y = \dfrac{a - bx}{c}$ is not a steady state, because both these curves lie at N_x.

For positive parameters, the system (9), (10) has always three following equilibrium points

$$(0,0), \quad \left(0, \frac{d}{f}\right), \quad \left(\frac{a}{b}, 0\right),$$

and one equilibrium

$$(x^*, y^*) = \left(\frac{af - cd}{bf - ce}, \frac{bd - ae}{bf - ce}\right) \quad \Longleftarrow \left\{ \begin{array}{l} a - bx - cy = 0 \\ d - ex - fy = 0 \end{array} \right.$$

(x^*, y^*) lies inside of M, if the terms $af - cd$, $bd - ae$ and $bf - ce$ are nonzero and have the same sign.

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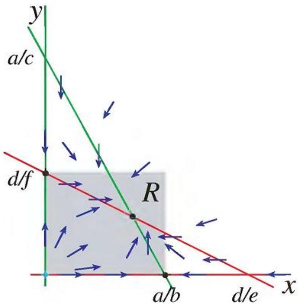
Let $s := \operatorname{sgn}(af - cd) = \operatorname{sgn}(bd - ae) = 1$, then because x^* and y^* are nonnegative, also $bf - ce > 0$ and the point (x^*, y^*) lies inside of M. In this case, the nullclines divide the phase space M into four parts.



In figure, there is depicted the vector field for $s = 1$.
If $x > \frac{a}{b}$, then
$\dot{x} = x(a - bx - cy) < x(a - a - cy) < 0$,
i.e., $\dot{x} < 0$ and x monotonically decrease and all initial conditions on the right-hand side of the line $\{(x, y) : x = \frac{a}{b}\}$ are moving to the left.

Similarly, if $y > \frac{d}{f}$, then $\dot{y} < 0$ and y is monotonically decreasing.

Consequence: The rectangle $R = \{(x, y), 0 \le x \le \frac{a}{b}, 0 \le y \le \frac{d}{f}\}$ is an invariant set, i.e., all trajectories that start in R stay in R. All initial conditions that lie in $M \setminus R$, must also necessarily end up in R.

Limit behavior for $t \longrightarrow \infty$ for any initial condition: the trajectory ends in one of the steady state points.

★ Phase portrait

Let us go back to two dimensional dynamical system (8)

$$\dot{z} = f(z) = \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} P(x,y) \\ Q(x,y) \end{pmatrix} .$$

If we ignore the time dependency, it is sometimes possible to find a solution as a parametrization of the curves in the phase space.

Idea: Let the trajectory is locally a graph of the function $y = Y(x)$. we obtain the following differential equation

$$\frac{\mathrm{d}Y}{\mathrm{d}x} = \frac{\dot{y}}{\dot{x}} = \frac{Q(x,Y)}{P(x,Y)} = F(x,y) .$$

This is one differential equation of the first order for the function $Y(x)$ (Y(x) . . . phase curve). This equation is usually non-autonomous, because a new vector field $F(x,Y)$ depends on a new independent variable x.

★ Example

Example The system

$$\begin{aligned} \dot{x} &= \mathrm{e}^{x+y}(x+y) \\ \dot{y} &= \mathrm{e}^{x+y}(x-y) \end{aligned}$$

can't be for $x(t), y(t)$ solved explicitly, but the equation for the phase curve is relatively simple:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x-y}{x+y}\,.$$

This equation can't be solved by separation, but if we define a new variable as $z = \dfrac{y}{x} = \dfrac{1}{x}y$ we obtain

$$\frac{\mathrm{d}z}{\mathrm{d}x} = -\frac{1}{x^2}y + \frac{1}{x}\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{x}\left(\frac{x-y}{x+y} - \frac{y}{x}\right) = \frac{1}{x}\left(\frac{1-z}{1+z} - z\right) = -\frac{1}{x}\cdot\frac{(z+1)^2-2}{1+z}\,.$$

This equation is already a separable differential equation of the first order.

We separate the variables and compute antiderivatives:

$$\int \frac{1+z}{(z+1)^2-2}\mathrm{d}z = -\int \frac{1}{x}\mathrm{d}x\,,$$

$$\frac{1}{2}\ln|(z+1)^2-2| = -\ln|x| + \ln C, \quad C>0,\ x>0,\ z=\frac{y}{x}\geq 0\,.$$

Let us denote

$O_1 = \{(x,z), x>0, z>-1+\sqrt{2}\}$ and $O_2 = \{(x,z), x>0, 0\leq z\leq -1+\sqrt{2}\}$.

Then in O_1: $z^2+2z-1 = \dfrac{C^2}{x^2}$, in O_2: $-z^2-2z-1+2 = \dfrac{C^2}{x^2}$.

After the back substitution for z we have two branches of the solution $y_{1,2} = -x \pm \sqrt{2x^2+C^2}$, i.e., the assumption that the graph $y = Y(x)$ is a graph of a function is wrong.

From the computations we see that the trajectories are hyperboles $(y+x)^2 - 2x^2 = \tilde{C}^2$.

Let us note that we got no information about the time dependence of trajectories.

Dynamical systems

A dynamical system. . . an evolutionary rule that defines the trajectory as a function of one parameter $t \in \mathbb{R}$ (time) on the phase (state) state M (typically $M = \mathbb{R}^n$). State of the system in time t depends also on the state in which the system was in time $t = 0$. Let us denote $\varphi(t, \sigma)$, $t \in \mathbb{R}$, $\sigma \in M$, the state of the system in time t, if in time $t = 0$ the system was in he state σ.

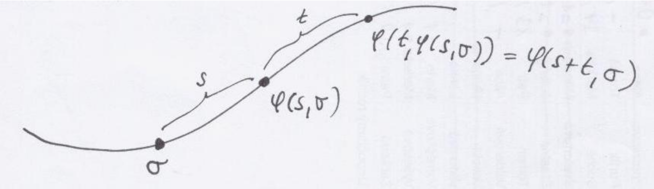
We obtain the mapping

$$\varphi : \mathbb{R} \times M \longrightarrow M \quad (\varphi(t, \sigma) \in M)\,,$$

while requesting

1. $\varphi(0, \sigma) = \sigma$, i.e., in time $t = 0$ the system was in he state σ.
2. the aditivity of time holds:

$$\varphi(t, \varphi(s, \sigma)) = \varphi(t + s, \sigma) \quad \forall t, s \in \mathbb{R},\ \forall \sigma \in M\,.$$



Definition The pair $\{S, \varphi\}$, where φ satisfies 1. and 2. items is called a continuous dynamical system on the state space S. The mapping φ is called dynamics of the system

Convention In what follows we will assume that the state space (phase space) $S \subseteq \mathbb{R}^n$ is an open set in $\mathbb{R}^n$ and $\mathbf{x} = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ is the state of the system.

For fixed $\mathbf{x} \in \mathbb{R}^n$ we set $\varphi_{\mathbf{x}}(t) = \varphi(t, \mathbf{x})\ \forall t \in \mathbb{R}$, and for fixed $t \in \mathbb{R}$ we set $\varphi^t(\mathbf{x}) = \varphi(t, \mathbf{x})\ \forall \mathbf{x} \in \mathbb{R}^n$.

We have two mappings

$$\underbrace{\varphi_{\mathbf{x}} : \mathbb{R} \longrightarrow \mathbb{R}^n}$$

parametrization of the curve passing through the point $\mathbf{x}$,

$$\varphi_{\mathbf{x}}(0) = \varphi(0, \mathbf{x}) = \mathbf{x} \quad \forall \mathbf{x} \in \mathbb{R}^n$$

$$\underbrace{\varphi^t : \mathbb{R}^n \longrightarrow \mathbb{R}^n}$$

fulfils the property 1. and 2. $\Longrightarrow$

(i) $\varphi^0(\mathbf{x}) = \varphi(0, \mathbf{x}) = \mathbf{x} \quad \forall \mathbf{x} \in \mathbb{R}^n$

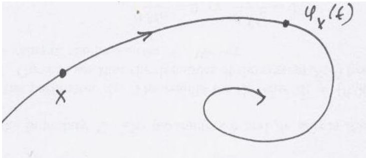
neboli $\varphi^0 = \mathrm{id}\big|_{\mathbb{R}^n}$

(ii) $\varphi^t \circ \varphi^s = \varphi^{t+s} \quad \Longrightarrow$

the mapping φ^t is simple (injective), because there exists an inverse mapping $\left(\varphi^t\right)^{-1} = \varphi^{-t}$

i.e., we have $\varphi^{-t} \circ \varphi^t = \varphi^0 = \mathrm{id}$

Time evolution of the system

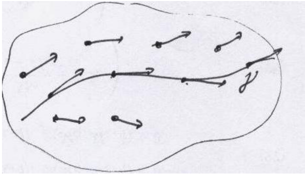


A point $\mathbf{x}$ is moving along the curve $\{\varphi_{\mathbf{x}}(t), t \in \mathbb{R}\}$, the motion is determined by the vector field $\vec{v}(\mathbf{x})$ on the state space $\mathbb{R}^n$, i.e., the vector field

$$\vec{v}(\mathbf{x}) = (v_1(\mathbf{x}), v_2(\mathbf{x}), \dots, v_n(\mathbf{x}))$$

is moving the state point (initial state) $\mathbf{x} = (x_1, \dots, x_n)$ along the curve $\{\varphi_{\mathbf{x}}(t), t \in \mathbb{R}\}$.

The curve γ represents the time development of he system, tangent vectors to the curve γ are equal to the corresponding vectors of the vector field $\vec{v}$.



For a given vector field $\vec{v}$ on the state space $\mathbb{R}^n$ we are looking for such curves (and their parametrizations) that have the tangent vectors equal to the vectors of the given vector field.

We have

$$\varphi_{\mathbf{x}}(t) = \mathbf{x}(t) = \underbrace{(x_1(t), x_2(t), \ldots, x_n(t))}, \quad t \in \mathbb{R}\,.$$

$n-$tuple of functions that define parametrically the curve γ

The tangent vector to the curve γ at $\mathbf{x}(t)$ is the vector

$$\overrightarrow{\mathbf{x}}'(t) = (x_1'(t), x_2'(t), \ldots, x_n'(t))\,.$$

We obtain the system of ordinary differential equations (SODE) in vector form:

$$\overrightarrow{\mathbf{x}}'(t) = \overrightarrow{\mathbf{v}}(\mathbf{x}(t))\,, \quad \text{usually we write only} \quad \overrightarrow{\mathbf{x}}' = \overrightarrow{\mathbf{v}}(\mathbf{x})\,. \tag{11}$$

If we rewrite the equation (11) in components, we obtain the system of n ordinary differential equations of the first order

$$\begin{aligned}
x_1'(t) &= v_1(x_1(t), x_2(t), \ldots, x_n(t)) \\
x_2'(t) &= v_2(x_1(t), x_2(t), \ldots, x_n(t)) \\
&\vdots \\
x_n'(t) &= v_n(x_1(t), x_2(t), \ldots, x_n(t))
\end{aligned}$$

Definition of trajectory and solution of SODE

Definition An ordered $n-$tuple of functions

$$\mathbf{x}(t) = (x_1(t), x_2(t), \ldots, x_n(t)), \quad t \in I \subset \mathbb{R},$$

is called a solution of the system (11) of ordinary differential equations of the first order. The curve parametrized by the solution $\mathbf{x}(t)$, is called the trajectory of the solution (trajecory of the system). Usually it is denoted by γ or $\gamma_{\mathbf{x}}$, if it is important that the trajectory goes through the point $\mathbf{x} \in \mathbb{R}^n$.

If $\mathbf{x}(t)$ is a solution of the system, $t_0 \in I$, $\mathbf{x}_0 \in \mathbb{R}^n$, then $\mathbf{x}(t_0) = \mathbf{x}_0$ is called the initial condition for the solution $\mathbf{x}(t)$. The corresponding trajectory is denoted as $\gamma_{\mathbf{x}_0}$.

The set of all trajectories of the system is so called phase portrait of the system.

★

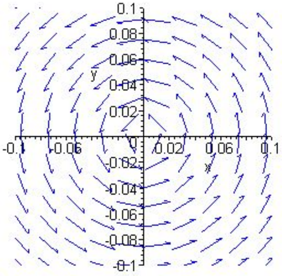
Example

$$\begin{aligned} x' &= -y \\ y' &= x \end{aligned} \implies \vec{\mathbf{v}}(x,y) = (-y, x)$$

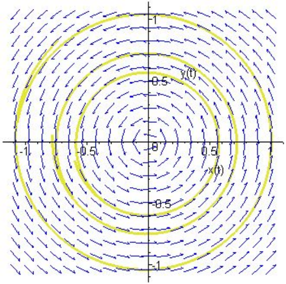
Solution

$$\begin{aligned} x(t) &= r\cos(t) \\ y(t) &= r\sin(t) \end{aligned} \qquad r \geq 0,\ t \in \mathbb{R}$$

$(x(t), y(t))\ldots$ parametrization of the circle with the center $S = [0,0]$ and with the radius r. Trajectories are concentric circles.



Vector field



Phase portrait

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Example Let us have the system

$$\begin{aligned} x' &= -\frac{y}{\sqrt{1+x^2+y^2}} \\ y' &= \frac{x}{\sqrt{1+x^2+y^2}} \end{aligned} \quad \Longrightarrow \quad \vec{\mathbf{w}}(x,y) = \frac{1}{\sqrt{1+x^2+y^2}}(-y,x)\,.$$

Hence,

$$\vec{\mathbf{w}}(x,y) = \frac{1}{\sqrt{1+x^2+y^2}} \cdot \vec{\mathbf{v}}\,,$$

i.e., directions of the vector field $\vec{\mathbf{w}}$ are the same as the directions of vectors in the field $\vec{\mathbf{v}}$, and thus vectors $\vec{\mathbf{w}}$ are tangent to concentric circles.

So, although we are not able to solve the system, we know its phase portrait.

Phase flow of the system

Let us have the system (11): $\vec{\mathbf{x}}'(t) = \vec{\mathbf{v}}(\mathbf{x}(t))$.

The mapping $\varphi : \mathbb{R} \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$, for which

(i) $\varphi(0, \mathbf{x}) = \mathbf{x} \quad \forall \mathbf{x} \in \mathbb{R}^n$,

(ii) $\varphi(t, \varphi(s, \mathbf{x})) = \varphi(t + s, \mathbf{x}) \quad \forall t, s \in \mathbb{R}$,

(iii) $\dfrac{\mathrm{d}\varphi_{\mathbf{x}}(t)}{\mathrm{d}t} = \vec{\mathbf{v}}(\varphi_{\mathbf{x}}(t)) \quad \forall t \in \mathbb{R},\ \forall \mathbf{x} \in \mathbb{R}^n$,

is called phase flow of the system (11), or phase flow of he vector field $\vec{\mathbf{v}}$.

Renarks

1. $\varphi(t, \mathbf{x}) = \varphi_{\mathbf{x}}(t)$ for fixed $\mathbf{x}$ is a solution of the system (11) with trajectory $\gamma_{\mathbf{x}}$. $\varphi_{\mathbf{x}}(t)$ fulfils initial condition $\varphi_{\mathbf{x}}(0) = \mathbf{x}$, i.e., $\mathbf{x}$ is the starting point of the trajectory $\gamma_{\mathbf{x}}$.
2. The phase flow $\varphi : \mathbb{R} \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ represents the set of all solutions of the system (11). By selecting different initial states $\mathbf{x} \in \mathbb{R}^n$ in $\varphi(t, \mathbf{x})$, we obtain different solutions of the system (11).

One-point trajectory

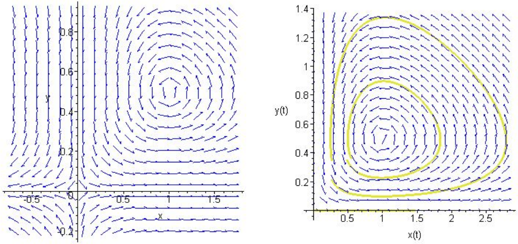
Let us solve again the system (11), i.e., $\vec{\mathbf{x}}'(t) = \vec{\mathbf{v}}(\mathbf{x}(t))$. Let $\varphi(t, \mathbf{x})$ be the phase flow of this system.

1. If $\mathbf{x}_0$ is an equilibrium of the system (11) then a constant mapping $\varphi_{\mathbf{x}_0} : \mathbb{R} \longrightarrow \mathbb{R}^n$ defined as $\varphi_{\mathbf{x}_0} = \mathbf{x}_0 \quad \forall t \in \mathbb{R}$ is a stationary solution of the system (11) with a one-point trajectory $\gamma_{\mathbf{x}_0} = \{\mathbf{x}_0\}$.

Example Find equilibria of the system

$$\begin{aligned} x' &= x - 2xy \\ y' &= -y + xy \end{aligned} \implies \begin{aligned} x - 2xy = 0 &\iff x = 0 \vee y = \frac{1}{2} \\ -y + xy = 0 &\iff x = 1 \vee y = 0 \end{aligned}$$

The system has two equilibria $S_1 = [0, 0]$, $S_2 = [1, 0.5]$. The picture shows the vector field and the phase portrait plotted in Maple. What is wrong?



Closed trajectory

2. closed trajectory: If there exist $\mathbf{x}_0 \in \mathbb{R}^n$ and $T > 0$ such that

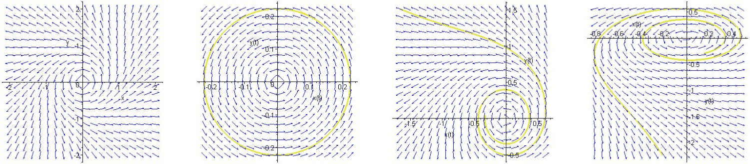
$$\varphi(0, \mathbf{x}_0) = \varphi(T, \mathbf{x}_0) = \mathbf{x}_0 \quad \text{a} \quad \varphi(t, \mathbf{x}_0) \neq \mathbf{x}_0 \; \forall t \in (0, T)\,,$$

then the trajectory $\gamma_{\mathbf{x}_0}$ is closed or periodic with the period T.

Example Let us consider the following nonlinear system of two differential equations:

$$\begin{aligned} x' &= -y + 0,5x(x^2 + y^2) \\ y' &= x + 0,5y(x^2 + y^2) \end{aligned}$$

The picture shows the vector field and the phase portrait for initial conditions: $x(0) = 0.1, y(0) = 2.3$ (closed trajectory), $x(0) = 0.6, y(0) = 0.3$ and $x(0) = -0.6, y(0) = 0.3$.



Trajectory is entering into resp. is leaving the equilibrium

3. Trajectory is entering into resp. is leaving the equilibrium state

Let $\mathbf{x}_1$ be an equilibrium of the system (11), $\gamma_{\mathbf{a}}$ be the trajectory of the solution $\varphi_{\mathbf{a}}(t)$, for which it holds:

If

(i) $\lim\limits_{t\to\infty} \varphi_{\mathbf{a}}(t) = \mathbf{x}_1$,

(ii) $\lim\limits_{t\to\infty} \overrightarrow{\varphi}'_{\mathbf{a}}(t) = \overrightarrow{\tau}_1$,

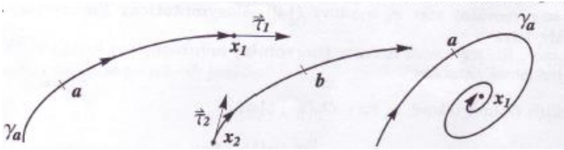
we say that the trajectory $\gamma_{\mathbf{a}}$ corresponding to the solution $\varphi_a(t)$ is entering into the equilibrium $\mathbf{x}_1$ in the direction of the vector $\overrightarrow{\tau}_1$.

Similarly, if

(i) $\lim\limits_{t\to-\infty} \varphi_{\mathbf{b}}(t) = \mathbf{x}_2$,

(ii) $\lim\limits_{t\to-\infty} \overrightarrow{\varphi}'_{\mathbf{b}}(t) = \overrightarrow{\tau}_2$,

we say that the trajectory $\gamma_{\mathbf{b}}$ corresponding to the solution $\varphi_b(t)$ is leaving the equilibrium $\mathbf{x}_2$ in the direction of the vector $\overrightarrow{\tau}_2$.



If the trajectory γ_a fulfils only (i) $\lim_{t\to\infty} \varphi_{\mathbf{a}}(t) = \mathbf{x}_1$ and the limit of φ'_a doesn't exist, we say that the trajectory "finishes"in $\mathbf{x}_1$. In this case, the trajectory approaches the equilibrium in a "spiral", i.e., it is not approaching the equilibrium in a particular direction.
If the trajectory γ_b fulfils only (i) and the limit of φ'_b doesn't exist, we say that the trajectory "starts"in $\mathbf{x}_2$.

4. heteroclinics, homoclinics
Let $\mathbf{x}_1, \mathbf{x}_2$ are two different equilibria of the system $\vec{\mathbf{x}}'(t) = \vec{\mathbf{v}}(\mathbf{x}(t))$. The trajectory γ that starts in the equilibrium $\mathbf{x}_1$ and finishes in the equilibrium $\mathbf{x}_2$ is called the heteroclinic trajectory.
Trajectory γ that starts in the equilibrium $\mathbf{x}_1$ and finishes also in the equilibrium $\mathbf{x}_1$ is called homoclinic trajectory.

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