Mathematics for chemical engineers

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8. Linear systems of ordinary differential equations

Obsah

A few notes to start

Linear systems of differential equations arise as models of systems in which the input is directly proportional to output.

Linear mapping

Let U, V be linear spaces. The mapping $\mathcal{L} : U \longrightarrow V$ is linear $\Longleftrightarrow$

1. $\mathcal{L}(u+v) = \mathcal{L}(u) + \mathcal{L}(v) \quad \forall u, v \in U$,
2. $L(cu) = c \cdot L(u) \quad \forall u \in U \, \forall c \in \mathbb{R}$.

Example Let the mapping $\mathcal{L} : \mathcal{C}^2(I) \longrightarrow \mathcal{C}(I)$ be given as $\mathcal{L}(f) = \alpha f'' + \beta f' + \gamma f$. $\mathcal{L}$ is a differential operator of the second order. Show that it is linear.

1. $f, g \in \mathcal{C}^2(I)$, $\mathcal{L}(f+g) = \alpha(f+g)'' + \beta(f+g)' + \gamma(f+g) = \alpha(f''+g'') + \beta(f'+g') + \gamma(f+g) = \alpha f'' + \beta f' + \gamma f + \alpha g'' + \beta g' + \gamma g = \mathcal{L}(f) + \mathcal{L}(g)$.
2. $f \in \mathcal{C}^2(I), c \in \mathbb{R}$, $\mathcal{L}(cf) = (cf)'' + (cf)' + (cf) = cf'' + cf' + cf = c(f'' + f' + f) = c \cdot \mathcal{L}(f)$.

Exercise Similarly show that the mapping $\mathcal{L} : U \longrightarrow V$ defined as $\mathcal{L}(f) = \int_a^b f \mathrm{d}x$ is linear. What are in this case spaces U and V?

Linear mapping in finite dimension

Theorem The mapping $\mathcal{L} : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is linear $\Longleftrightarrow$ it is represented by a matrix $\mathbf{A}_{m\times n}$,, i.e., $\mathcal{L}\mathbf{x} = \mathbf{A}\mathbf{x}$, $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{A}\mathbf{x} \in \mathbb{R}^m$.

Application to a system of linear differential equations

The phase space is usually a vector space where unknowns are usually real (physical quantities) $\Rightarrow$ we will assume that the phase space is $\mathbb{R}^n$. Let us note that if $\mathbf{f} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a linear mapping then

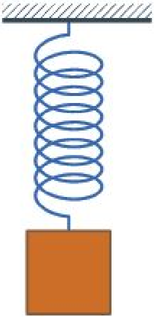
$$f_i(\mathbf{x}) = \sum_{j=1}^{n} a_{ij} x_j \quad \text{for } i = 1, \ldots, n, \quad \mathbf{A} = \{a_{ij}\}_{i,j=1}^{n}, \ \text{t.j.} \ \mathbf{A}_{n\times n} .$$

Then the vector field is given in a matrix form: $\mathbf{f}(\mathbf{x}) = \mathbf{A}\mathbf{x}$, $\mathbf{x} \in \mathbb{R}^n$, and the differential equation reads as

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{A}\mathbf{x} .$$

★ Harmonic oscilator

Example The harmonic oscillator model the motion of matter attached to the spring. The (linear) force $F = -k(x - L)$ pulls the mass, L is the spring length at equilibrium, k is a material coefficient.



Newton's law for the spring has the form $m\ddot{x} = F = -k(x - L)$ ...affine differential equation of the second order. It is linear only in the case $kL = 0$. Nevertheless we can linearize the equation by substracting equilibrium $x^* = L$: Let $\xi = x - x^*$ be the deviation from equilibrium. Then $\dot{\xi} = \dot{x}$ and $\ddot{\xi} = \ddot{x}$. We obtain

$$m\ddot{\xi} = -k(x - x^*) \implies \ddot{\xi} = -\frac{k}{m} \cdot \xi.$$

The linear differential equation of the second order

$$\ddot{\xi} = -\frac{k}{m} \cdot \xi. \tag{1}$$

can be rewritten as a system of two differential equations of the first order:

$$\begin{aligned} \dot{\xi} &= \eta \\ \dot{\eta} &= \ddot{\xi} = -\frac{k}{m}\xi \end{aligned} \quad \text{i.e.,} \quad \frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & 0 \end{pmatrix} \cdot \begin{pmatrix} \xi \\ \eta \end{pmatrix}. \tag{2}$$

Exercise Solve the equation (1) as a linear differential equation of the second order and the equation (2) as a system of two differential equations of the first order. Compare solutions.

Remark: Eigenvalues of the matrix in (2):

$$\det\begin{pmatrix} -\lambda & 1 \\ -\frac{k}{m} & -\lambda \end{pmatrix} = \lambda^2 + \frac{k}{m} \Longrightarrow \lambda_{1,2} = \pm\mathrm{i}\sqrt{\frac{k}{m}}.$$

Eigenvalues and eigenvectors of the matrix

λ is an eigenvalue of the matrixe $\mathbf{A}_{n\times n}$ and $\mathbf{x} \neq \mathbf{0}$ is a corresponding eigenvector $\iff$ $\mathbf{Ax} = \lambda\mathbf{x}$ $\iff$ $(\mathbf{A} - \lambda\mathbf{E})\mathbf{x} = \mathbf{0}$ $\implies$ $(\mathbf{A} - \lambda\mathbf{E})$ is singular.

We obtain the characteristic equation (polynomial of the degree n with the coefficient 1 for λ^n):

$$\det(\mathbf{A} - \lambda\mathbf{E}) = \mathbf{0}\,. \tag{3}$$

Zeros of the characteristic equation are eigenvalues of the matrix $\mathbf{A}$. From he equation $(\mathbf{A} - \lambda\mathbf{E})\mathbf{x} = \mathbf{0}$ we obtain, after the substitution of the particular computed eigenvalue, a correspnding eigenvector. An eigenvalue may have more eigenvectors.

Remark If $\mathbf{x} \neq \mathbf{0}$ is an eigenvector then also $c\mathbf{x}$ is an eigenvector for all $c \in \mathbb{R} \setminus \{0\}$. As a consequence, the vector "length"can be chosen arbitrarily.

The Fundamental Theorem of Algebra: The polynomial of nth degree has n zeros (zeros are counted with their algebraic multiplicities).

The algebraic multiplicity of the eigenvalue λ: If $p(r) = (r - \lambda)^k q(r)$ and $q(\lambda) \neq 0$, then λ is a zero of p with the algebraic multiplicity k. If $k > 1$, we say that λ is a multiple eigenvalue.

Geometric multiplicity of the eigenvalue λ

Let $\mathbf{A}_{n\times n},\ \lambda$ is the eigenvalue of the matrix $\mathbf{A}$.

The geometric multiplicity of the eigenvalue λ is defined as the dimension of the null-space of the matrix $\mathbf{B} = \mathbf{A} - \lambda\mathbf{E}$. If $h(\mathbf{B})$ is the rank of the matrix $\mathbf{B}$ then the dimension of the null-space of the matrix $\mathbf{B}$ is $\dim \mathbf{B} = n - h(\mathbf{B})$.
Notation:

$$\underbrace{\dim \mathcal{N}(\mathbf{A} - \lambda\mathbf{E})}_{\text{nullity of the matrix } (\mathbf{A} - \lambda\mathbf{E})} = \text{the dimension of the null-space of the matrix } \mathbf{A} - \lambda\mathbf{E}\,.$$

Remark

$$0 \leq \text{ geometric multiplicity of } \lambda \underbrace{\leq}_{\text{If the inequality is sharp } < \ldots \text{deficit of eigenvectors}} \text{ algebraic multiplicity of } \lambda. \tag{4}$$

★ Example: the eigenvalue has two LI eigenvectors

Example $\mathbf{A} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \Rightarrow \mathbf{A} - \lambda\mathbf{E} = \begin{pmatrix} 2-\lambda & 0 \\ 0 & 2-\lambda \end{pmatrix}$, $\lambda = 2$ is the double eigenvalue, more precisely, algebraic multiplicity of the eigenvalue $\lambda = 2$ is two. Let us calculate the eigenvectors:

$$(\mathbf{A} - 2\mathbf{E})\mathbf{h} = \mathbf{0}, \text{ t.j. } \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \ \mathbf{h} = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

i.e., to the double eigenvalue $\lambda = 2$ correspond two linearly independent eigenvectors

$$\mathbf{h}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{h}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

We say that to the eigenvalue $\lambda = 2$ corresponds a complete system of eigenvectors. Here, the complete system consists of the vectors $\mathbf{h}_1$ and $\mathbf{h}_2$. What about the geometric multiplicity?

$$\mathbf{A} - 2\mathbf{E} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \Longrightarrow \dim\mathcal{N}(\mathbf{A} - 2\mathbf{E}) = 2 - 0 = 2,$$

i.e., also the geometric multiplicity of the eigenvalue $\lambda = 2$ is 2.

★ Example: deficit of eigenvectors

Example $\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \Rightarrow \mathbf{A} - \lambda\mathbf{E} = \begin{pmatrix} 2-\lambda & 1 \\ 0 & 2-\lambda \end{pmatrix}$, algebraic multiplicity of he eigenvalue $\lambda = 2$ is 2. Let us calculate the eigenvectors.

$$(\mathbf{A} - 2\mathbf{E})\mathbf{h} = \mathbf{0}, \text{ t.j. } \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \ \mathbf{h} = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

i.e., to the doubled eigenvalue $\lambda = 2$ corresponds only one eigenvector. What is the geometric multiplicity?

$$\mathbf{A} - 2\mathbf{E} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \Longrightarrow \dim\mathcal{N}(\mathbf{A} - 2\mathbf{E}) = 2 - 1 = 1, \quad \text{i.e.,}$$

$$\text{geometric multiplicity} = 1 \ < \ \text{algebraic multiplicity} = 2\,.$$

In this case we say that the matrix **A** has the deficit of eigenvectors.

What does it mean ?

We say that the matrix

$\mathbf{A}_{n\times n}$ has the deficit of eigenvectors if its eigenvectors do not form the basis of $\mathbb{R}^n$,

i.e., there is less eigenvectors than n. The deficit of eigenvectors always occurs if at least for one eigenvalue λ of the matrix $\mathbf{A}$ is the algebraic multiplicity λ sharply greater than its geometric multiplicity, see (4).

The system of the eigenvectors can be supplemented to the base of $\mathbb{R}^n$ using so called

generalized eigenvectors.

Remark If the matrix has a deficit of eigenvectors it is not diagonalizable.

Exercise Show that eigenvectors form a linearly independent system of vectors.

Generalized eigenvectors

The vector **k** is a generalized eigenvector of the order r that corresponds to the eigenvalue $\lambda \iff$

$$\begin{aligned} (\mathbf{A}-\lambda\mathbf{E})^r\,\mathbf{k} &= \mathbf{0} \\ (\mathbf{A}-\lambda\mathbf{E})^{r-1}\,\mathbf{k} &\neq \mathbf{0}\,. \end{aligned}$$

Remark The eigenvector is a generalized eigenvector of the first order, because

$$(\mathbf{A}-\lambda\mathbf{E})\,\mathbf{k} = \mathbf{0} \quad \text{a} \quad \mathbf{k} \neq \mathbf{0}\,.$$

If **k** is a generalized eigenvector of order r, we define vectors $\mathbf{h}_1, \ldots \mathbf{h}_r$ as follows:

$$\begin{aligned} \mathbf{h}_r &= (\mathbf{A}-\lambda\mathbf{E})^0\mathbf{k} = \mathbf{k}, \\ \mathbf{h}_{r-1} &= (\mathbf{A}-\lambda\mathbf{E})^1\mathbf{k}, \\ &\vdots \\ \mathbf{h}_1 &= (\mathbf{A}-\lambda\mathbf{E})^{r-1}\mathbf{k}\,. \end{aligned}$$

Linearly independent (show it) vectors $\mathbf{h}_1, \ldots, \mathbf{h}_r$ form a chain of generalized eigenvectors of length r .

Remark The vector $\mathbf{h}_1$ is the eigenvector because

$$\mathbf{h}_1 \neq \mathbf{0} \quad \text{a} \quad (\mathbf{A}-\lambda\mathbf{E})\mathbf{h}_1 = (\mathbf{A}-\lambda\mathbf{E})^r\mathbf{k} = \mathbf{0}\,.$$

★ Example: the deficit of eigenvectors continued

Let us go back to the example of the matrix $\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$. The algebraic multiplicity of the eigenvalue $\lambda = 2$ is 2, the geometric multiplicity is 1, i.e. to the eigenvalue λ corresponds only one eigenvector $\mathbf{h} = (1, 0)^{\mathrm{T}}$. To obtain a base of $\mathbb{R}^2$ we need to find one generalized eigenvector $\mathbf{k}$. It can be found as a solution of the nonhomogeneous linear system of algebraic equations

$$(\mathbf{A} - 2\mathbf{E})\mathbf{k} = \mathbf{h}, \quad \mathbf{k} = (k_1, k_2)^{\mathrm{T}} \neq \mathbf{0}, \quad \text{i.e.}$$

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Longrightarrow \begin{pmatrix} k_2 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

So $k_2 = 1$, $k_1 \in \mathbb{R}$ is arbitrary, for simplicity we can choose $k_1 = 0$.

Let us show that $\mathbf{k} = (0, 1)^{\mathrm{T}}$ is a generalized eigenvector of order 2:

$$(\mathbf{A} - 2\mathbf{E})^2\mathbf{k} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

$$(\mathbf{A} - 2\mathbf{E})\mathbf{k} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

System of two linear differential equations of the 1st order

Let $n = 2$ and let us consider a linear autonomous system. Let $\mathbf{A}$ be the matrix of the system

$$\mathbf{z}'(t) = \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \mathbf{A}\mathbf{z}(t)\,, \qquad \mathbf{z}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}\,. \tag{5}$$

Let the fundamental system consist of two functions $\{\mathbf{z}_1(t), \mathbf{z}_2(t)\}$. Then all solutions of the system have the form

$$\mathbf{z}(t) = C_1\mathbf{z}_1(t) + C_2\mathbf{z}_2(t), \quad C_1, C_2 \in \mathbb{R}\,,\ t \in \mathbb{R}\,.$$

Thus the set of all solutions is a linear space with the dimension $\dim = 2$. The null element of this space is a stationary solution of the system, i.e. $x(t) \equiv 0, y(t) \equiv 0\,.$

Which functions form the fundamental system ?

Fundamental system consists of two functions

Let $n = 2$. We seek a solution of the linear autonomous system (5) in the form $\mathbf{z}(t) = \mathrm{e}^{\lambda t}\mathbf{h}$, where λ is an eigenvalue of the matrix $\mathbf{A}$, $\mathbf{h} \neq \mathbf{0}$ is the corresponding eigenvector. Thus λ is a root of the characteristic equation (3). We have

$$\det(\mathbf{A} - \lambda\mathbf{E}) = \mathbf{0} \iff \lambda^2 - \operatorname{tr}(\mathbf{A}) + \det\mathbf{A} = 0.$$

For $n = 2$ the eigenvalues are roots of the quadratic equation. We will discuss three particular cases of roots.

I. $\lambda_1 \neq \lambda_2$ are two different real eigenvalues $\Longrightarrow$ f.s.= $\{\mathrm{e}^{\lambda_1 t}\mathbf{h}_1, \mathrm{e}^{\lambda_2 t}\mathbf{h}_2\}$ and a general solution is

$$\mathbf{z}(t) = C_1\mathrm{e}^{\lambda_1 t}\mathbf{h}_1 + C_2\mathrm{e}^{\lambda_2 t}\mathbf{h}_2\,, \quad C_1, C_2 \in \mathbb{R},\ t \in \mathbb{R}\,.$$

In general for $n > 2$, $\mathbf{A}_{n\times n}$, if the characteristic equation has n different real roots $\lambda_1, \ldots, \lambda_n$, $\mathbf{h}_i$ are the corresponding eigenvectors then the general solution has the form

$$\mathbf{z}(t) = C_1\mathrm{e}^{\lambda_1 t}\mathbf{h}_1 + C_2\mathrm{e}^{\lambda_2 t}\mathbf{h}_2 + \cdots + C_n\mathrm{e}^{\lambda_n t}\mathbf{h}_n\,, \quad C_1, C_2 \in \mathbb{R},\ t \in \mathbb{R}\,.$$

II. $\lambda_{1,2} = a \pm \mathrm{i}b\,,\, b \neq 0\ \ldots$ two complex conjugate eigenvalues

Because the eigenvalues are imaginary $\implies$ also the corresponding eigenvectors are imaginary $\implies$ also the fundamental system is imaginary and each solution is imaginary.

But the fundamental system is the base of the two-dimensional space of all solutions. This space has infinitely many bases, so we can choose the real one.

In particular, if $\mathbf{h}_1 = \mathbf{u} + \mathrm{i}\mathbf{v}$ is the imaginary eigenvector that belongs to the imaginary eigenvalue $\lambda_1 = a + \mathrm{i}b$, then the fundamental system consists of two real vector functions $\mathbf{z}_1(t) = \mathrm{e}^{at}\mathbf{u}$, $\mathbf{z}_2(t) = \mathrm{e}^{at}\mathbf{v}$.

Thus, the general solution has the form

$$\mathbf{z}(t) = C_1\mathrm{e}^{at}\mathbf{u} + C_2\mathrm{e}^{at}\mathbf{v}\,, \quad C_1, C_2 \in \mathbb{R},\ t \in \mathbb{R}\,.$$

In general for $n > 2$, $\mathbf{A}_{n\times n}$, if among the roots of the characteristic equation are two complex conjugate eigenvalues then the corresponding functions of the fundamental system have always the form $\mathrm{e}^{\Re\lambda\cdot t}\Re\mathbf{h},\ \mathrm{e}^{\Re\lambda\cdot t}\Im\mathbf{h}$, where h is the corresponding imaginary eigenvector.

Double eigenvalue

III. Characteristic equation has one double real root λ.
The first vector function of the fundamental system will be $\mathbf{z}_1(t) = e^{\lambda t}\mathbf{h}$, where $\mathbf{h} \neq \mathbf{0}$ is the corresponding eigenvector to λ. But there is still lack of another eigenvector, the second function of the fundamental system. This second solution can be find using generalized eigenvector $\mathbf{k}$, $\mathbf{k} \neq \mathbf{0}$. Generalized eigenvector $\mathbf{k}$ is a solution od the nonhomogeneous system of linear algebraic equations:

$$(\mathbf{A} - \lambda\mathbf{E})\mathbf{k} = \mathbf{h}\,.$$

The second function of the fundamental system is then

$$\mathbf{z}_2(t) = e^{\lambda t}(t \cdot \mathbf{h} + \mathbf{k})\,. \tag{6}$$

Exercise Show that if $\lambda \in \mathbb{R}$ is a double eigenvalue then functions $\mathbf{z}_1$, $\mathbf{z}_2$ form a linearly independent system of vector functions such that each of these functions is a solution of the system (5), i.e., they form a fundamental system of the solutions of (5).

System of n linear differential equations

Now, let us have a system of linear differential equations of the first order ,

$$\mathbf{z}'(t) = \mathbf{A}\mathbf{z}(t), \quad \mathbf{A}_{n\times n}\,, \tag{7}$$

and let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ be eigenvectors of the matrix $\mathbf{A}$. Then each solution of the system (7) can be written as

$$\mathbf{w} = \sum_{i=1}^{n} C_i \mathbf{v}_i, \quad \mathbf{C} = (C_1, C_2, \ldots, C_n) \in \mathbb{R}^n\,, \text{ t.j. } \mathbf{w} \in \underbrace{\mathrm{span}}_{\text{linear hull}}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}\,.$$

If $\mathrm{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\} = \mathbb{R}^n$, we say that $\mathbf{A}$ has a complete system of eigenvectors, i.e., eigenvectors form a basis of $\mathbb{R}^n$.
If we assemble a matrix from the eigenvectors ,

$$\mathbf{P} = \begin{pmatrix} | & & | \\ \mathbf{v}_1 & \ldots & \mathbf{v}_n \\ | & & | \end{pmatrix}\,, \tag{8}$$

then $\mathbf{P}$ is regular, i.e., $\det(\mathbf{P}) \neq 0$, and as a consequence the inverze $\mathbf{P}^{-1}$ exists.

★ Example, $n = 3$

Example $\mathbf{A} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \implies$ eigenvalues are $\lambda_{1,2} = 1,\ \lambda_3 = 2$.

Algebraic multiplicity of $\lambda = 1$ is 2, λ_3 has algebraic multiplicity 1.
At first, let us compute eigenvectors corresponding to $\lambda = 1$:

$$(\mathbf{A} - 1 \cdot \mathbf{E})\,\mathbf{h} = \mathbf{0}\,,\ \text{t.j.} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The rank of the matrix $\mathbf{A} - \mathbf{E}$ is 2, $n = 3 \implies \dim\mathcal{N}(\mathbf{A} - \mathbf{E}) = 1$. The matrix has a deficit of eigenvectors, because the geometric multiplicity of the eigenvalue $\lambda = 1$ less then its algebraic multiplicity. We have to calculate one eigenvector $\mathbf{h}$ and one generalized eigenvector $\mathbf{k}$, that correspond to the double eigenvalue $\lambda = 1$.

Check that $\mathcal{N}(\mathbf{A} - \mathbf{E}) = \{\mathbf{h} \in \mathbb{R}^3, \mathbf{h} = (t, 0, 0)^{\mathrm{T}}, t \in \mathbb{R}\}$.

As the eigenvector choose $\mathbf{h} = (1, 0, 0)^{\mathrm{T}}$.

★

The generalized eigenvector is a solution of the nonhomogeneous system of linear algebraic equations:

$$(\mathbf{A}-\lambda\mathbf{E})\mathbf{k}=\mathbf{h} \implies \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

Check that the generalized eigenvector corresponding to the eigenvalue $\lambda = 1$ is $\mathbf{k} = (1,1,0)^{\mathrm{T}}$.

For the eigenvalue $\lambda_3 = 2$:

$$\mathbf{A}-2\cdot\mathbf{E} = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \implies \dim\mathcal{N}(\mathbf{A}-2\mathbf{E}) = 3-2 = 1\,.$$

The geometric multiplicity of $\lambda_3 = 2$ is equal to its algebraic multiplicity $= 1$. In this case the whole chain of generalized eigenvectors consists only of the eigenvector $\mathbf{h}_3 = (0,0,1)^{\mathrm{T}}$.

★ What is it good for in solving systems of differential equations?

Let us consider a system of three differential equations with the matrix of the system **A**:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}\,.$$

We already know that the matrix has eigenvalue $\lambda_{1,2} = 1$ with the geometric multiplication 1 and algebraic multiplicity 2, the corresponding eigenvector **h** and generalized eigenvector **k**. The eigenvalue $\lambda_3 = 2$ has both geometric and algebraic multiplicity 1. the corresponding eigenvector is $\mathbf{h}_3$. The fundamental system is $\mathrm{f.s.} = \{\mathbf{x}_1(t), \mathbf{x}_2(t), \mathbf{x}_3(t)\}$, where

$$\mathbf{x}_1(t) = \mathrm{e}^t \cdot \mathbf{h}, \quad \mathbf{x}_2(t) = \mathrm{e}^t(t \cdot \mathbf{h} + \mathbf{k}), \quad \mathbf{x}_3(t) = \mathrm{e}^{2t} \cdot \mathbf{h}_3\,.$$

For $\mathbf{x}_2$, see (6) . The general solution is

$$\mathbf{x}(t) = C_1\mathrm{e}^t \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + C_2\mathrm{e}^t \left(t \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right) + C_3\mathrm{e}^{2t} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

$t \in \mathbb{R},\ C_i \in \mathbb{R},\ i = 1, 2, 3.$

★

Let us assemble from the vectors $\mathbf{h}, \mathbf{k}$ and $\mathbf{h}_3$ the matrix

$\mathbf{P} = (\mathbf{h}, \mathbf{k}, \mathbf{h}_3) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, see (8).

Check that $\mathbf{P}$ is a regular matrix and that $\mathbf{P}^{-1} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

Now we will compute

$$\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 2 \end{array}\right).$$

The last matrix is so called Jordan canonical form of the matrix $\mathbf{A}$. In our case it consists from two Jordan blocks. The first block corresponds to the eigenvalue $\lambda = 1$. The geometric multiplicity of $\lambda = 1$ is 1 $\Rightarrow$ it is only one block, the algebraic multiplicity is 2, i.e. the size of the block is 2×2. On the diagonal of this block there is the eigenvalue 1 and above the diagonal in the column that corresponds to the generalized eigenvector $\mathbf{k}$, there is also 1. Below the diagonal there are zeros. The second block corresponds to the eigenvalue $\lambda = 2$. Here algebraic multiplicity = geometric multiplicity = 1 $\Longrightarrow$ one block of the size 1×1 with the eigenvalue 2 on the "diagonal".

★ Jordan canonical form

For any matrix $\mathbf{A}_{n\times n}$ there exists a nonsingular matrix $\mathbf{P}$ such that

$$\mathbf{J} := \mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathrm{diag}(\mathbf{C}_1, \mathbf{C}_2, \dots, \mathbf{C}_p), \quad \text{where} \quad \mathbf{C}_j = \lambda_j \mathbf{E} + \mathbf{R}_j, \quad \mathbf{C}_j \in \mathbb{R}^{n_j \times n_j},$$

$$\mathbf{R}_j = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ & & \ddots & \ddots & \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix},$$

λ_j, $j = 1, \dots, p$, are, in general not different, eigenvalues of the matrix $\mathbf{A}$.

$\mathbf{J}$ ... Jordan canonical form of the matrix $\mathbf{A}_{n\times n}$.

For the matrix $\mathbf{A}$ it holds $\mathbf{A} = \mathbf{P}\mathbf{J}\mathbf{P}^{-1}$, i.e., $\mathbf{A} \sim \mathbf{J}$, the matrices $\mathbf{A}$ and $\mathbf{J}$ are similar, they have the same eigenvalues, but the matrix $\mathbf{J}$ has much simpler structure. It is nearly diagonal: it has eigenvalues on the diagonal, above the diagonal, in columns that corresponds to the generalized eigenvectors, there are 1 and everywhere else there are zeros.

We mentioned the general form just for completeness. We will investigate mainly systems of two or three differential equations.

Theorem Let $n = 2$, $\mathbf{A}_{2\times 2}$ be a square real matrix.
Then there exists a regular transformation matrix $\mathbf{S}_{2\times 2}$ (dependent on eigenvalues λ_1, λ_2 of the matrix $\mathbf{A}$) such that the matrix $\mathbf{B} = \mathbf{S}^{-1}\mathbf{A}\mathbf{S}$ has one of the following forms:

(i) both λ_1 and λ_2 have the algebraic and geometric multiplicity equal 1, i.e.,

$$\lambda_1 \neq \lambda_2,\ \lambda_1, \lambda_2 \in \mathbb{R} \quad \Longrightarrow \quad \mathbf{B} = \left(\begin{array}{c|c} \lambda_1 & 0 \\ \hline 0 & \lambda_2 \end{array}\right),$$

(to each eigenvalue corresponds on the diagonal of $\mathbf{B}$ one block of the size 1×1)

(ii) λ_1, λ_2 are imaginary and
algebraic multiplicity = geometric multiplicity = 1, i.e.,

$$\lambda_{1,2} = a \pm \mathrm{i}b,\ b \neq 0 \quad \Longrightarrow \quad \mathbf{B} = \left(\begin{array}{c|c} a & -b \\ \hline b & a \end{array}\right),$$

(iii) **A** has one double eigenvalue $\lambda_0 \in \mathbb{R}$ such that its algebraic and geometric multiplicity is 2, to λ_0 correspond two linearly independent eigenvectors, i.e.,

$$\lambda_0 \in \mathbb{R},\ \lambda_0 \text{ correspond two LI eigenvectors} \quad \Longrightarrow \quad \mathbf{B} = \begin{pmatrix} \lambda_0 & 0 \\ 0 & \lambda_0 \end{pmatrix},$$

(two block, 1×1 each)

(iv) **A** has one double eigenvalue $\lambda_0 \in \mathbb{R}$, such hat its algebraic multiplicity is 2, geometric multiplicity is 1, λ_0 corresponds one eigenvector and one generalized eigenvector, i.e.

$\lambda_0 \in \mathbb{R}$, λ_0 corresponds one eigenvector and one generalized eigenvector

$$\Longrightarrow \quad \mathbf{B} = \begin{pmatrix} \lambda_0 & 1 \\ 0 & \lambda_0 \end{pmatrix},$$

(one block 2×2) .

Two-dimensional linear systems with constant coefficients

Let $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Let us denote $\varrho(\lambda) = \det(\mathbf{A} - \lambda\mathbf{E})$ its characteristic polynomial. Then the characteristic equation of the matrix **A** is $\varrho(\lambda) = \lambda^2 - \tau\lambda + \delta = 0$, where $\tau = \mathrm{tr}(\mathbf{A}) = a + d$ is the trace of the matrix **A**, $\delta = \det(\mathbf{A}) = ad - bc$ is the determinant of **A**. Thus, eigenvalues depend on the trace $tr(\mathbf{A})$ of the matrix **A** and on the determinant $\det(\mathbf{A})$ of the matrix **A**.

Roots of $\varrho(\lambda)$:

$$\lambda_{\pm} = \frac{\tau \pm \sqrt{D}}{2}, \quad \text{where } D = \tau^2 - 4\delta = a^2 + 2ad + d^2 - 4ad + 4bc = (a-d)^2 + 4bc.$$

In the plane $\tau - D$ $(\mathrm{tr}(\mathbf{A}) - \det(\mathbf{A}))$ there are 5 different "domains"of eigenvalues.

Remark D ... discriminant. If $D > 0$ there are two different real eigenvalues, if $D < 0$ there are two imaginary complex conjugate eigenvalues.

Transformation matrix

Let us have the system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$, $\mathbf{A}_{2\times 2}$, and let **S** be a transformation matrix, $\mathbf{B} = \mathbf{S}^{-1}\mathbf{A}\mathbf{S}$. Matrices **A** and **B** are similar, they have the same characteristic polynomial and, as a consequence also the same eigenvalues. We obtain

$$\mathbf{A} = \mathbf{S}\mathbf{B}\mathbf{S}^{-1} \quad \Longrightarrow \quad \dot{\mathbf{x}} = \mathbf{S}\mathbf{B}\mathbf{S}^{-1}\mathbf{x}\,.$$

We substitute $\mathbf{y} := \mathbf{S}^{-1}\mathbf{x}$, i.e., $\mathbf{x} = \mathbf{S}\mathbf{y}$ and $\dot{\mathbf{x}} = \mathbf{S}\dot{\mathbf{y}}$. Then

$$\mathbf{S}\dot{\mathbf{y}} = \mathbf{S}\mathbf{B}\mathbf{y} \quad \Longrightarrow \quad \underbrace{\dot{\mathbf{y}} = \mathbf{B}\mathbf{y}\,.}_{\text{the system after the transformation}}$$

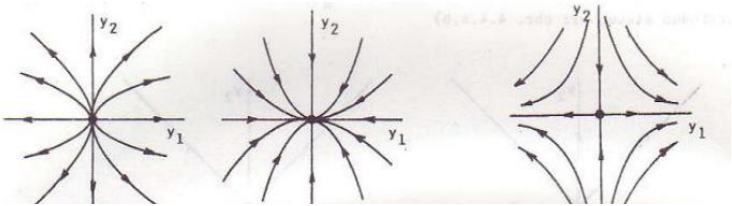
The aim of the transformation:
To simplify the system of differential equations to so called
canonical form
of the system of differential equations.

Remark The following sketches of phase portraits are taken from the textbook A. Klíč, M. Kubíček: Matematika III - Diferenciální rovnice, VŠCHT Praha, 1992, ISBN 80-7080-162-X.

Phase portraits of systems in canonical forms

Theorem Let $n = 2$, $\mathbf{A}_{2\times 2}$ be a real square matrix.
Then there exists a regular matrix $\mathbf{S}$ such that the matrix $\mathbf{B} = \mathbf{S}^{-1}\mathbf{AS}$ has one of the following four forms (in the last column there is the corresponding name of the equilibrium):

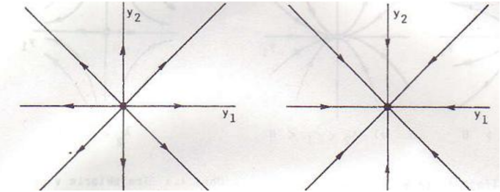
$$\text{(i)} \quad \lambda_1 \neq \lambda_2, \lambda_1, \lambda_2 \in \mathbb{R}, \quad \mathbf{B} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \quad \text{node, saddle}$$



$$\underbrace{\lambda_1 > 0, \lambda_2 > 0}_{\text{unstable node}} \qquad \underbrace{\lambda_1 < 0, \lambda_2 < 0}_{\text{stable node}} \qquad \underbrace{\lambda_1 \cdot \lambda_2 < 0}_{\text{saddle (unstable)}}$$

Into the saddle two trajectories enter and two trajectories come out. These trajectories are called saddle separatrix.

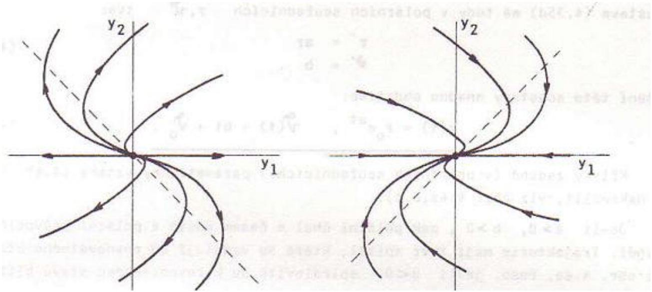
(ii) $\lambda_1 = \lambda_2 := \lambda_0 \in \mathbb{R}, \quad \mathbf{B} = \begin{pmatrix} \lambda_0 & 0 \\ 0 & \lambda_0 \end{pmatrix}$ dicritical node



$\underbrace{\lambda_0 > 0}$ unstable dicritical node $\qquad$ $\underbrace{\lambda_0 < 0}$ stable dicritical node

Trajectories are rays that come from ($\lambda_0 > 0$), resp. enter into ($\lambda_0 < 0$) the equilibrium.

$$\text{(iii)} \quad \lambda_1 = \lambda_2 := \lambda_0 \in \mathbb{R}, \quad \mathbf{B} = \begin{pmatrix} \lambda_0 & 1 \\ 0 & \lambda_0 \end{pmatrix}$$ Jordan node

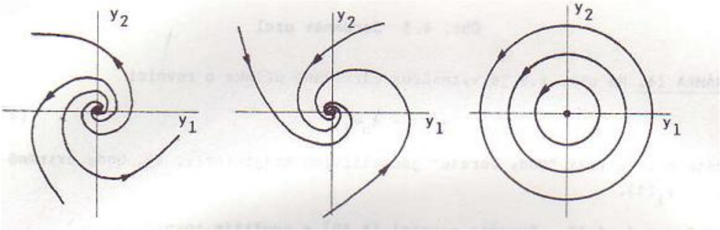


$\lambda_0 > 0$
unstable Jordan node

$\lambda_0 < 0$
stable Jordan node

Dashed line has the equation $y_2 = -\lambda_0 y_1$. On this line there are "turning points" of the individual trajectories, i.e., points of extremes of the function $y_1(t)$.

(iv) $\lambda_{1,2} = a \pm \mathrm{i}b, b \neq 0$ $\mathbf{B} = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ focus, center



$\underbrace{a > 0}$ unstable focus $\underbrace{a < 0}$ stable focus $\underbrace{a = 0}$ center

If $a > 0$, the trajectories have a spiral shape, that depart from the equilibrium, if $a < 0$, the trajectories are spiraling toward the equilibrium.
If $a = 0$, the trajectories are circles centered at the origin, i.e., closed trajectories that correspond to the periodic solutions.

The characteristic equation for the matrix **A** is

$$\lambda^2 - \mathrm{tr}(\mathbf{A})\lambda + \det(\mathbf{A}) = 0 \quad \Longrightarrow \quad \lambda_{1,2} = \frac{\mathrm{tr}(\mathbf{A}) \pm \sqrt{\mathrm{tr}^2(\mathbf{A}) - 4\det(\mathbf{A})}}{2}, \quad \text{t.j.}$$

$$\begin{aligned}
\lambda_1 + \lambda_2 &= \frac{1}{2}\left(\mathrm{tr}(\mathbf{A}) + \sqrt{\mathrm{tr}^2(\mathbf{A}) - 4\det(\mathbf{A})} + \mathrm{tr}(\mathbf{A}) - \sqrt{\mathrm{tr}^2(\mathbf{A}) - 4\det(\mathbf{A})}\right) = \\
&= \mathrm{tr}(\mathbf{A}) \\
\lambda_1 \cdot \lambda_2 &= \frac{1}{4}\left(\mathrm{tr}(\mathbf{A}) + \sqrt{\mathrm{tr}^2(\mathbf{A}) - 4\det(\mathbf{A})}\right) \cdot \left(\mathrm{tr}(\mathbf{A}) - \sqrt{\mathrm{tr}^2(\mathbf{A}) - 4\det(\mathbf{A})}\right) = \\
&= \frac{1}{4}(\mathrm{tr}^2(\mathbf{A}) - \mathrm{tr}^2(\mathbf{A}) + 4\det(\mathbf{A})) = \det(\mathbf{A}).
\end{aligned}$$

We will classify phase portraits (resp. equilibria) in dependence on the trace of the matrix **A** and on the determinant of the matrix **A**.

Remark To each matrix **A** corresponds just one point in the $\mathrm{tr}(\mathbf{A}) - \det(\mathbf{A})$ plane. Conversely, to each point in the plane $\mathrm{tr}(\mathbf{A}) - \det(\mathbf{A})$ correspond infinitely many matrices.

Let us investigate the discriminant of the characteristic equation

$$D = \mathrm{tr}^2(\mathbf{A}) - 4\det(\mathbf{A}) \;\Rightarrow\; D = 0 \;\Rightarrow\; \det(\mathbf{A}) = \frac{1}{4}\mathrm{tr}^2(\mathbf{A}) \;\ldots \text{parabola}$$

I. $D > 0 \quad \Rightarrow \quad$ **A** has two different real eigenvalues and it holds

$$\det(\mathbf{A}) < \frac{1}{4}\mathrm{tr}^2\mathbf{A}\,.$$

(i) $\det(\mathbf{A}) < 0 \quad \Leftrightarrow \quad \lambda_1 \cdot \lambda_2 < 0 \Rightarrow$ equilibrium-type saddle

(ii) $\det(\mathbf{A}) > 0 \quad \Leftrightarrow \quad \lambda_1 \cdot \lambda_2 > 0 \Rightarrow$ equilibrium-type node

stable node: $\underbrace{\lambda_1 < 0, \lambda_2 < 0,}_{\lambda_1 + \lambda_2 < 0 \;\Rightarrow\; \mathrm{tr}(\mathbf{A}) < 0}$ unstable node: $\underbrace{\lambda_1 > 0, \lambda_2 > 0}_{\lambda_1 + \lambda_2 > 0 \;\Rightarrow\; \mathrm{tr}(\mathbf{A}) > 0}$

(iii) $\det(\mathbf{A}) = 0$, t.j. for example $\lambda_1 = 0, \lambda_2 \neq 0 \;\Rightarrow$

Phase portrait with lines of equilibrium states:

- $\mathrm{tr}(\mathbf{A}) = \lambda_2 < 0 \ldots$ trajectories enter into the equilibrium
- $\mathrm{tr}(\mathbf{A}) = \lambda_2 > 0 \ldots$ trajectories come out from the equilibrium

II. $D < 0 \quad \Rightarrow \quad$ **A** has two imaginary complex conjugate eigenvalues

$\lambda_{1,2} = a \pm \mathrm{i}b,\ b \neq 0$ and

$$\det(\mathbf{A}) > \frac{1}{4}\mathrm{tr}^2\mathbf{A}\,.$$

In this case $\mathrm{tr}(\mathbf{A}) = a + \mathrm{i}b + a - \mathrm{i}b = 2a$.

(i) $\mathrm{tr}(\mathbf{A}) < 0 \ \Leftrightarrow\ a < 0 \ \Rightarrow$ equilibrium-type focus, stable

(ii) $\mathrm{tr}(\mathbf{A}) > 0 \ \Leftrightarrow\ a > 0 \ \Rightarrow$ equilibrium-type focus, unstable

(iii) $\mathrm{tr}(\mathbf{A}) = 0, \ \Rightarrow\ a = 0 \ \Rightarrow\ \lambda_{1,2} = \pm \mathrm{i}b \ \Rightarrow$ equilibrium-type center

III. $D = 0 \ \Rightarrow$ **A** has one double eigenvalue λ_0 and

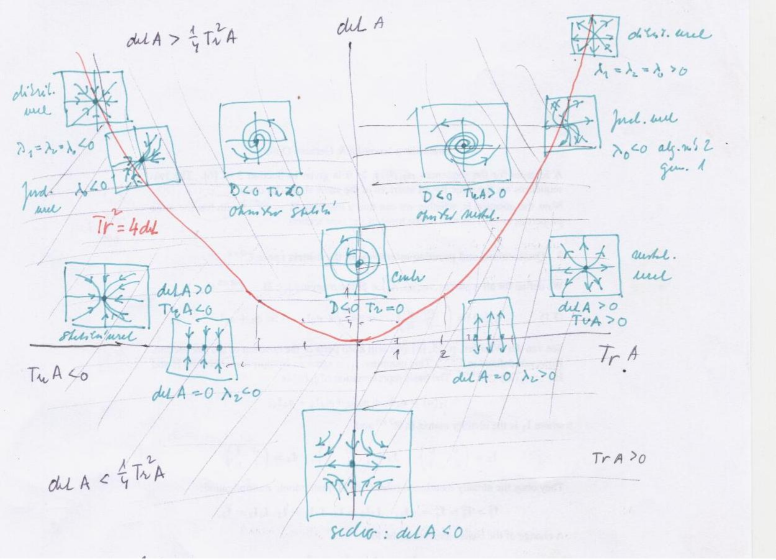
$$\det(\mathbf{A}) = \frac{1}{4}\mathrm{tr}^2\mathbf{A}\,.$$

(i) $\mathrm{tr}(\mathbf{A}) < 0 \ \Leftrightarrow\ \lambda_0 < 0 \ \Rightarrow$ equilibrium type dicritical node (Jordan node), stable

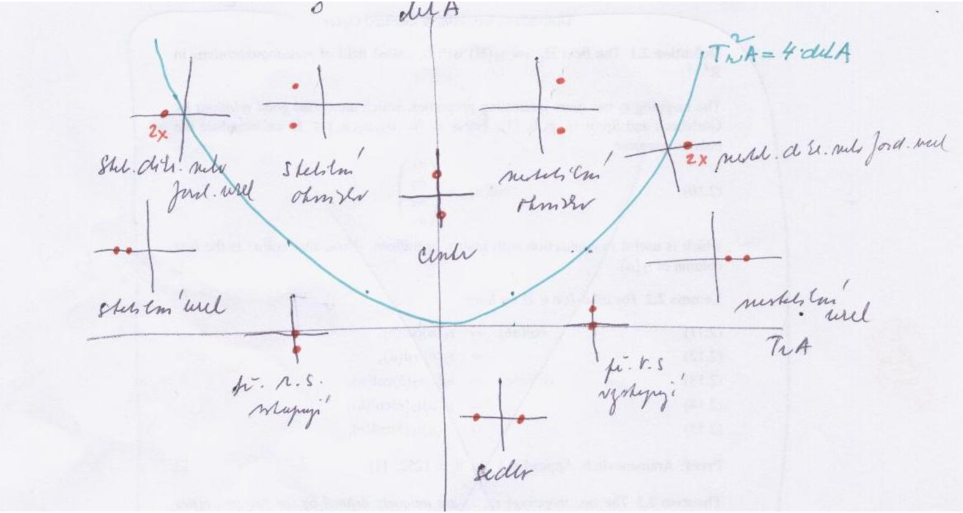
(ii) $\mathrm{tr}(\mathbf{A}) > 0 \ \Leftrightarrow\ \lambda_0 > 0 \ \Rightarrow$ equilibrium type dicritical node (Jordan node), unstable

(iii) $\mathrm{tr}(\mathbf{A}) = 0 \ \Rightarrow\ \lambda_0 = 0 \ \Rightarrow$ phase portrait with line of equilibrium states and the trajectories parallel with the line.

Classification of phase portraits in dependence on $\mathrm{tr}(\mathbf{A})$ and $\det(\mathbf{A})$



Classification of phase portraits in dependence on eigenvalues of (A)



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Example Let us solve the system

$$\begin{aligned} x_1' &= x_1 + x_2 \\ x_2' &= -5x_1 - x_2 \end{aligned}, \text{ t.j. } \mathbf{A} = \begin{pmatrix} 1 & 1 \\ -5 & -1 \end{pmatrix}, \quad \mathbf{A} - \lambda \mathbf{E} = \begin{pmatrix} 1-\lambda & 1 \\ -5 & -1-\lambda \end{pmatrix},$$

characteristic equation: $\lambda^2 + 4 = 0 \implies \lambda_{1,2} = \pm 2\mathrm{i}, \quad \mathbf{h} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \pm \mathrm{i} \begin{pmatrix} 0 \\ 2 \end{pmatrix}.$

Thus,

$$\mathbf{S} = \begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix}, \quad \mathbf{S}^{-1} = \begin{pmatrix} 1 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}, \quad \mathbf{B} := \mathbf{S}^{-1}\mathbf{A}\mathbf{S} = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}.$$

If we substitute $\mathbf{x} := \mathbf{S}\mathbf{y}$, we obtain the equivalent system of differential equations

$$\dot{\mathbf{y}} = \mathbf{B}\mathbf{y} \quad \text{i.e.} \quad \begin{aligned} \dot{y}_1 &= 2y_2 \\ \dot{y}_2 &= -2y_1 \end{aligned}, \text{ and again } \lambda_{1,2} = \pm 2\mathrm{i}\,.$$

Eigenvalues lies on the the imaginary axis and
$\det(\mathbf{A}) = 4,\ \mathrm{tr}(\mathbf{A}) = 0,\ D = -16 \quad \Rightarrow$ equilibrium-type center

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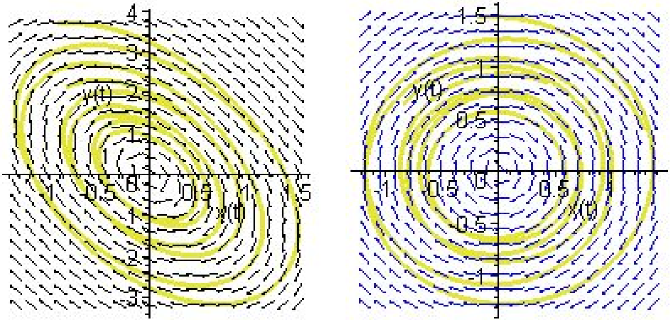


Phase portrait: on the left of the original system, on the right of the system after the transformation.

Exercise Show that the general solution of the original system is

$$\begin{aligned} x_1(t) &= C_1 \cos 2t + C_2 \sin 2t\,, \quad C_1, C_2 \in \mathbb{R},\ t \in \mathbb{R}, \\ x_2(t) &= (-C_1 + 2C_2)\cos 2t - (2C_1 + C_2)\sin 2t \end{aligned}$$

and compute the particular solution whose trajectory passes through the point $x_1(0) = 1,\ x_2(0) = 0$.

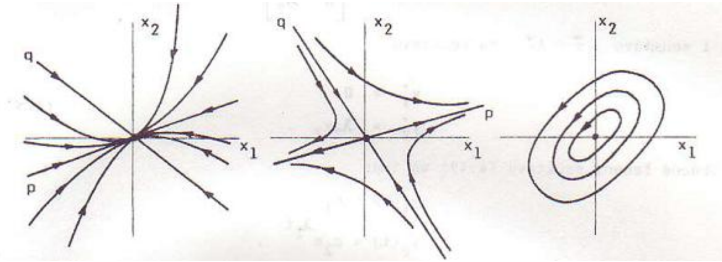
The reverse transformation - phase portraits in the plane $x_1 - x_2$

The reverse transformation: $\mathbf{x} := \mathbf{S}\mathbf{y}$

I. $\det(\mathbf{A}) \neq 0$

(a) $\mathbf{A}$ has two different real eigenvalues λ_1, λ_2 with the corresponding eigenvectors $\mathbf{h}_1, \mathbf{h}_2 \Rightarrow$ the transformation $\mathbf{x} := \mathbf{S}\mathbf{y}$ maps the axes y_1 and y_2 to the straight lines p, q with the directional vectors $\mathbf{h}_1, \mathbf{h}_2$. In the case of the node, trajectories enter (except two of them) the node in the direction of that straight line p, q which directional vector corresponds to eigenvalue with the smaller absolute value.

(b) The phase portrait of the dicritical node doesn't change after transformation because the linear transformation $\mathbf{x} = \mathbf{S}\mathbf{y}$ maps segments to segments.



stable node saddle (unstable) center

II. $\det(\mathbf{A}) = 0$. Then the characteristic equation has the form

$$\lambda^2 - \mathrm{tr}(\mathbf{A})\lambda = 0 \quad \Longrightarrow \quad \lambda_1 = 0,\ \lambda_2 = \mathrm{tr}(\mathbf{A})\,.$$

(a) $\lambda_2 \neq 0$

Let $\mathbf{r}, \mathbf{s}$ be eigenvectors corresponding to λ_1, λ_2. We obtain the equilibria of the system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ as solutions of the homogeneous system of linear equations

$$\mathbf{A}\mathbf{x} = \mathbf{0}\,.$$

Because $\det(\mathbf{A}) = 0$, the non-zero matrix $\mathbf{A}$ is singular, it has rank 1 and the set of all solutions is a one-dimensional subspace of $\mathbb{R}^2$, i.e., the straight line p passing through the origin in the direction of the eigenvector $\mathbf{r}$, that corresponds to the eigenvalue $\lambda_1 = 0$. The straight line p is called the straight line of equilibria, because each point of p is an equilibrium of the system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$.

The matrix $\mathbf{S} = (\mathbf{r}, \mathbf{s})$ transforms the matrix $\mathbf{A}$ into the Jordan canonical form

$$\underbrace{\mathbf{S}^{-1}\mathbf{A}\mathbf{S}}_{=\mathbf{B}} = \begin{pmatrix} 0 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

and the system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ to the system

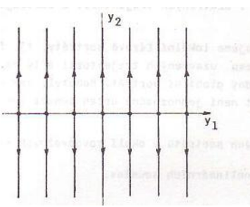
$$\begin{aligned} y_1' &= 0 \\ y_2' &= \lambda_2 y_2\,. \end{aligned}$$

The general solution of the system $\dot{\mathbf{y}} = \mathbf{B}\mathbf{y}$ has the form

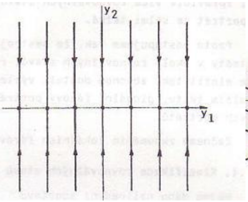
$$y_1(t) = C_1, \quad y_2(t) = C_2 e^{\lambda_2 t}.$$

These are parametric equations of trajectories and we can draw the phase portrait. The straight line of equilibria is y_1-axis, trajectories are segments of lines parallel to the axis y_2. After the reverse transformation the axis y_1 becomes the straight line of equilibria p with the directional vector $\mathbf{r}$,the axis y_2 becomes the straight line q with the directional vector $\mathbf{s}$.

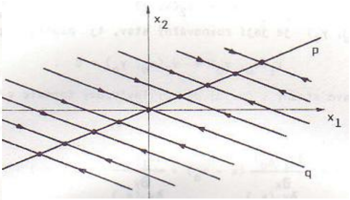
Remark The transformation $\mathbf{x} = \mathbf{S}\mathbf{y}$ preserves parallel lines.



$\det(\mathbf{A}) = 0, \ \lambda_2 > 0$



$\det(\mathbf{A}) = 0, \ \lambda_2 < 0$



Phase portrait in $x_1 - x_2$ plane.

(b) The matrix $\mathbf{A}$ has a double zero eigenvalue to which belong one eigenvector $\mathbf{h}$ and one generalized eigenvector $\mathbf{k}$.

Then the matrix $\mathbf{S} = (\mathbf{hk})$ transforms the matrix $\mathbf{A}$ to Jordan's canonical form

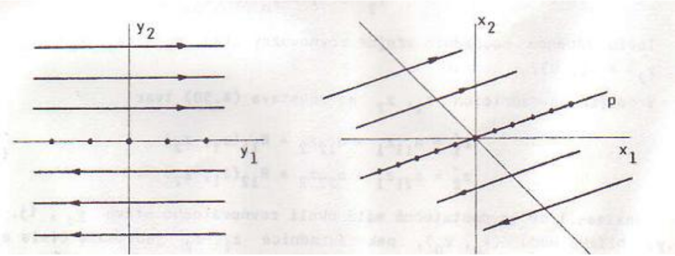
$$\mathbf{B} := \mathbf{S}^{-1}\mathbf{AS} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

and the system $\dot{\mathbf{x}} = \mathbf{Ax}$ to the system

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= 0\,. \end{aligned}$$

The general solution of the system $\dot{\mathbf{y}} = \mathbf{By}$ has the form

$$y_1(t) = C_1 + C_2 t, \quad y_2(t) = C_2\,.$$



The case of a double zero eigenvalue

Exercise

1. Classify the equilibrium and sketch the phase portrait of the system

$$\begin{aligned} x_1' &= x_1 - 5x_2 \\ x_2' &= x_1 - x_2 \end{aligned}.$$

2. Investigate the phase portrait of the system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$, where $\mathbf{A}$ is a zero matrix of order 2.

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