Mathematics for chemical engineers

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9. Phase portraits of non-linear systems of differential equations

Obsah

Introduction

Phase portraits of linear systems in the plane are global phase portraits - the system of trajectories covers the entire plane. For nonlinear systems we usually investigate the local phase portraits in the neighborhood of the equilibria and then we compose them, to obtain the final global phase portrait.

Remark The following sketches of phase portraits are taken from the textbook A. Klíč, M. Kubíček: Matematika III - Diferenciální rovnice, VŠCHT Praha, 1992, ISBN 80-7080-162-X.

Classification of equilibria of nonlinear systems

Let us have a nonlinear system

$$\begin{aligned}\dot{x} &= v_1(x,y)\\ \dot{y} &= v_2(x,y).\end{aligned}$$

Let $S_0 = (x_0, y_0)$ is an isolated equilibrium of this system, i.e., it holds

$$v_1(x_0,y_0) = v_2(x_0,y_0) = 0\,,$$

and there exists a neighborhood of the equilibrium S_0 such that there is no other equilibrium in this neighborhood.

Taylor expansion v_1, v_2 in (x_0, y_0), $(x,y) \in \mathcal{O}(x_0,y_0)$,

$$\begin{aligned}v_1(x,y) &= \underbrace{v_1(x_0,y_0)}_{=0} + \frac{\partial v_1(x_0,y_0)}{\partial x}(x-x_0) + \frac{\partial v_1(x_0,y_0)}{\partial y}(y-y_0) + R_1(x,y)\\ v_2(x,y) &= \underbrace{v_2(x_0,y_0)}_{=0} + \frac{\partial v_2(x_0,y_0)}{\partial x}(x-x_0) + \frac{\partial v_2(x_0,y_0)}{\partial y}(y-y_0) + R_2(x,y)\end{aligned}$$

Let us denote

$$a_{11} = \frac{\partial v_1(x_0, y_0)}{\partial x} \quad a_{12} = \frac{\partial v_1(x_0, y_0)}{\partial y}$$
$$a_{21} = \frac{\partial v_2(x_0, y_0)}{\partial x} \quad a_{22} = \frac{\partial v_2(x_0, y_0)}{\partial y}$$

We get the Jacobi matrix J at the point S_0, $S_0 = (x_0, y_0)$:

$$J(S_0) = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} \frac{\partial v_1(x_0, y_0)}{\partial x} & \frac{\partial v_1(x_0, y_0)}{\partial y} \\ \frac{\partial v_2(x_0, y_0)}{\partial x} & \frac{\partial v_2(x_0, y_0)}{\partial y} \end{pmatrix}$$

Let us introduce a transformation of coordinates $z_1 := x - x_0, \; z_2 := y - y_0$. Then

equilibrium $S_0 = (x_0, y_0)$ is transformed into the equilibrium $(z_1, z_2) = (0, 0)$.

We obtain the system

$$\begin{aligned} \dot{z}_1 &= a_{11} z_1 + a_{12} z_2 + R_1(z_1, z_2) \\ \dot{z}_2 &= a_{21} z_1 + a_{22} z_2 + R_2(z_1, z_2) \, . \end{aligned}$$

Linearization of a nonlinear system

If the neighborhood of S_0 is small enough, the numbers $z_1 = x - x_0,\ z_2 = y - y_0$ and reminders $R_1(z_1, z_2),\ R_2(z_1, z_2)$ will be very small, i.e.,

$$R_1 = \mathcal{O}(\underbrace{x - x_0}_{z_1})^2, \quad R_2 = \mathcal{O}(\underbrace{y - y_0}_{z_2})^2.$$

If we neglect the reminders we get the system of linear differential equations:

$$\begin{aligned} \dot{z}_1 &= a_{11}z_1 + a_{12}z_2 \\ \dot{z}_2 &= a_{21}z_1 + a_{22}z_2 \end{aligned} \implies \underbrace{\dot{\mathbf{z}} = \mathbf{J}(S_0) \cdot \mathbf{z}}$$

linearization of the nonlinear system in $\mathcal{O}(S_0)$.

$\dot{\mathbf{z}} = \mathbf{J}(S_0) \cdot \mathbf{z}$ ... the equation in variations of the system $\dot{x} = v_1(x, y),\ \dot{y} = v_2(x, y)$,
$\mathbf{J}(S_0)$ ... the matrix of linearization.

The phase portraits of systems of linear differential equations we already know, now we compute them only in a small neighborhood of the steady state.

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Example 1 Sketch the phase portrait of the system of nonlinear differential equations

$$\begin{aligned} \dot{x} &= \ln(y^2 - x) \\ \dot{y} &= x - y - 1\,. \end{aligned}$$

Solution At first, we find the equilibria (also called steady states) of the system

$$\begin{aligned} \ln(y^2 - x) &= 0 \\ x - y - 1 &= 0 \end{aligned} \quad \Longrightarrow \quad \begin{aligned} y^2 - x &= 1 \\ -y + x &= 1 \end{aligned} \quad \Longrightarrow$$

the system has two steady states $S_1 = (0, -1)$, $S_2 = (3, 2)$.

$$\mathbf{J}(x,y) = \begin{pmatrix} \dfrac{-1}{y^2 - x} & \dfrac{2y}{y^2 - x} \\ 1 & -1 \end{pmatrix} \Longrightarrow \mathbf{J}(S_1) = \begin{pmatrix} -1 & -2 \\ 1 & -1 \end{pmatrix}, \ \mathbf{J}(S_2) = \begin{pmatrix} -1 & 4 \\ 1 & -1 \end{pmatrix}.$$

The characteristic equation and eigenvalues of $\mathbf{J}(S_1)$:
$\lambda^2 + 2\lambda + 3 = 0 \Longrightarrow \lambda_{1,2} = -1 \pm \mathrm{i}\sqrt{2} \quad \Longrightarrow$ S_1 is a stable focus.

Similarly, the characteristic equation and eigenvalues of $\mathbf{J}(S_2)$:
$\lambda^2 + 2\lambda - 3 = 0 \Longrightarrow \lambda_1 = 1, \quad \lambda_2 = -3 \quad \Longrightarrow$ S_2 is a saddle (unstable).

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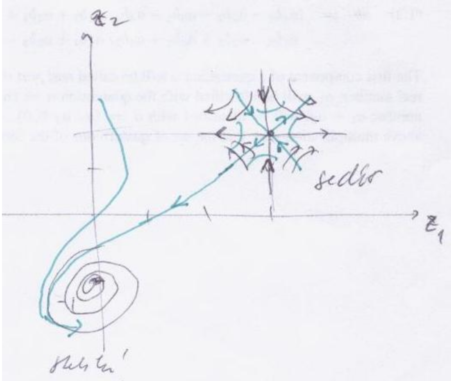
Remark To the eigenvalue $\lambda_1 = 1$ corresponds the eigenvector $\mathbf{h}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and to the eigenvalue $\lambda_2 = -3$ corresponds the eigenvector $\mathbf{h}_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$. The vectors $\mathbf{h}_1$ and $\mathbf{h}_2$ determine the directions in which for $\lambda > 0$ the separatrixs of the saddle come out from S_2. For $\lambda < 0$ the separatrixs enter the steady state S_2.



Theorem Let $S_0 = (x_0, y_0)$ be a steady state of the nonlinear system

$$\begin{aligned} \dot{x} &= \mathbf{v}_1(x, y), \\ \dot{y} &= \mathbf{v}_2(x, y). \end{aligned} \tag{1}$$

Let $\mathbf{J}(S_0)$ be a corresponding matrix of linearization and let both two eigenvalues of the matrix $\mathbf{J}$ have non-zero imaginary parts.
Then the phase portrait of the nonlinear system (1) in a certain neighborhood of the steady state S_0 is qualitatively the same as the phase portrait of the system

$$\dot{\mathbf{z}} = \mathbf{J}(S_0)\mathbf{z} \quad \text{in the neighbourhood of the origin}\,. \tag{2}$$

Definition Let $S_0 = (x_0, y_0)$ be an isolated steady state of the system (1) and $\mathbf{J}(S_0)$ be the corresponding matrix of linearisation with eigenvalues λ_1, λ_2, which don't lie on the imaginary axis. Then

1) If $\lambda_1 \cdot \lambda_2 > 0$, $\lambda_1, \lambda_2 \in \mathbb{R}$, we call the equilibrium S_0 node.

2) If $\lambda_1 \cdot \lambda_2 < 0$, $\lambda_1, \lambda_2 \in \mathbb{R}$, we call the equilibrium S_0 saddle.

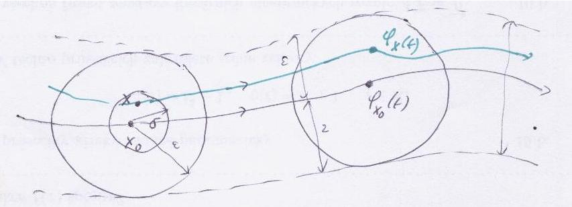
3) If $\lambda_{1,2} = a \pm ib$, $a \cdot b \neq 0$ we call the equilibrium S_0 focus.

Thus, except in the case λ_1, λ_2 are located on the imaginary axis, the classification of phase portraits of nonlinear systems in the neighborhood of the steady state can be converted into classification of phase portraits of the linearization of these systems in the neighborhood of the origin.

★ Lyapunov stability

Definition The steady state of the system $\dot{\mathbf{x}}(t) = \mathbf{v}(\mathbf{x}(t))$ is stable (in the sense of Lyapunov) $\iff$

$$\forall \mathcal{O}_\varepsilon(x_0)\ \exists \mathcal{O}_\delta(x_0)\ \text{ such that }\ \forall x \in \mathcal{O}_\delta(x_0)\ \text{ is }\ \varphi_x(t) \in \mathcal{O}_\varepsilon(\varphi_{x_0}(t))\ \forall t \geq 0\,.$$

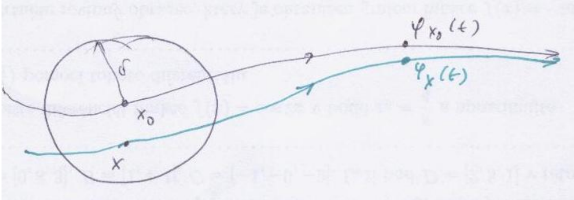


Remark In the case $n = 2$, Lyapunov stability means that any trajectory $\varphi_x(t)$, which starts at $\delta-$neighborhood of the point x_0 remains inside the tube with a maximum radius ε for all $t \geq 0$.

★ Asymptotic stability

Definition The steady state of the system $\dot{\mathbf{x}}(t) = \mathbf{v}(\mathbf{x}(t))$ is asymptotically stable $\iff$

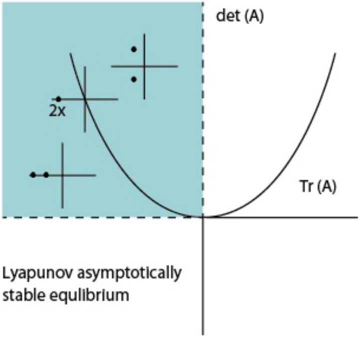
- it is stable (in the sense of Lyapunov) and
- $\lim_{t\to\infty} \rho(\varphi_{x_0}(t) - \varphi_x(t)) = 0 \quad \forall x \in \mathcal{O}_\delta(x_0)$



Remark In the case $n = 2$, asymptotic stability means that any trajectory $\varphi_x(t)$, which starts at $\delta-$neighborhood of the point x_0 gradually converge to $\varphi_{x_0}(t)$ with increasing t.

★ Stability of steady states

Theorem Let S_0 be an equilibrium of the system (1) and let $\mathbf{J}(S_0)$ be a corresponding matrix of linearization.



If both eigenvalues of the matrix $\mathbf{J}(S_0)$ have negative real part, then S_0 is asymptotically Lyapunov stable equilibrium.

If there exists an eigenvalue of the matrix $\mathbf{J}(S_0)$ with a positive real part, the equilibrium S_0 is Lyapunov unstable.

Let us go back to Example 1.
$S_1 = (0,1)$, $\mathbf{J}(S_1)$ has eigenvalues $-1 \pm i\sqrt{2} \Rightarrow$ S_1 is asymptotically Lyapunov stable

$S_2 = (3,2)$, $\mathbf{J}(S_2)$ has eigenvalues $\lambda_1 = 1, \lambda_2 = -3 \Rightarrow$ S_2 is Lyapunov unstable.

★ Example

Example Determine the values of the parameters a, b for which the zero solution of the system is asymptotically stable.

$$\dot{x} = ax + y\,, \quad \dot{y} = x + by\,.$$

Solution We calculate the eigenvalues λ_i of the coefficient matrix $\mathbf{A}$:

$$\mathbf{A} = \begin{bmatrix} a & 1 \\ 1 & b \end{bmatrix}, \quad \det(\mathbf{A} - \lambda\mathbf{E}) = 0 \;\Rightarrow$$

$$\begin{vmatrix} a-\lambda & 1 \\ 1 & b-\lambda \end{vmatrix} = 0 \;\Rightarrow\; \lambda^2 - (a+b)\lambda + ab - 1 = 0\,.$$

Solve the resulting quadratic equation with the parameters a, b.

$$D = (a+b)^2 - 4(ab-1) = a^2 + 2ab + b^2 - 4ab + 4 = (a-b)^2 + 4 > 0\,.$$

As can be seen, the discriminant is always positive. Therefore, the eigenvalues are real numbers and are defined by

$$\lambda_{1,2} = \frac{a + b \pm \sqrt{(a-b)^2 + 4}}{2}\,.$$

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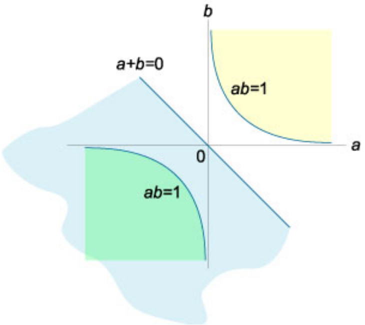
We find the set of values of the numbers a, b at which the eigenvalues λ_1, λ_2 are negative (this means that the system is asymptotically stable):

$$\lambda_1 < 0 \qquad a + b + \sqrt{(a-b)^2 + 4} < 0$$
$$\lambda_2 < 0 \qquad a + b - \sqrt{(a-b)^2 + 4} < 0$$



Adding the two inequalities, we get $a + b < 0$. In this case, the second inequality $\sqrt{(a-b)^2 + 4} > a + b$ holds for all a, b satisfying $a + b < 0$. Now we solve the first inequality and obtain $ab > 0$ for all a, b satisfying $a + b < 0$. The common solution is the region (shaded in green) below the hyperbola $ab = 1$ in the left half-plane. For all values of a, b from this region, the solution of the system will be asymptotically stable.

The Hartman-Grobman and Poincaré-Bendixon Theorems are two of the most powerful tools used in dynamical systems.

The Hartman-Grobman theorem allows us to represent the local phase portrait about certain types of equilibria in a nonlinear system by a similar, simpler linear system that we can find by computing the system's Jacobi matrix at the equilibrium point.

The Poincaré-Bendixon theorem tells us that if we can show that an orbit with an initial condition in a region is contained in that region for all future time then there must be a closed orbit or a fixed point in the region.

Homeomorfismus

Let us consider two systems of nonlinear differential equations

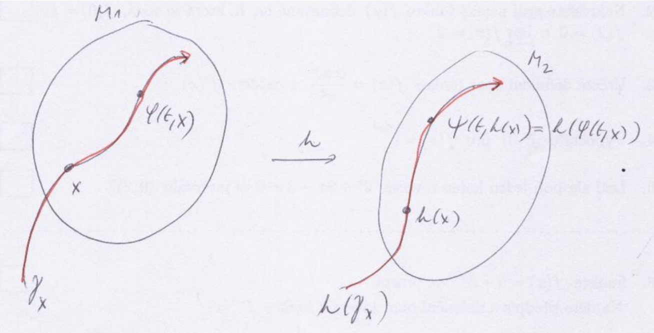
$$\dot{\mathbf{x}} = \mathbf{v}(\mathbf{x}),\ \mathbf{x} \in M_1 \subseteq \mathbb{R}^2,\ \varphi(t, \mathbf{x}) \text{ the phase flow on } M_1, \quad (3)$$
$$\dot{\mathbf{x}} = \mathbf{u}(\mathbf{x}),\ \mathbf{x} \in M_2 \subseteq \mathbb{R}^2,\ \psi(t, \mathbf{x}) \text{ the phase flow on } M_2. \quad (4)$$

Definition We say that phase portraits of systems (3) and (4) are topologically equivalent, if there exists a homeomorphism $h : M_1 \longrightarrow M_2$, which maps the trajectories of the first system to the trajectories of the second system while maintaining orientation, i.e., it holds

$$h(\varphi(t, \mathbf{x})) = \psi(t, h(\mathbf{x}))\,.$$

Remark $h : M_1 \longrightarrow M_2$ is a homeomorphism $\Leftrightarrow$ h is a continuous bijection (one-to-one and onto function) with a continuous inverse (denoted h^{-1}). The existence of homeomorphism tells us that M_1 and M_2 have analogous structures.

Remark $h : M_1 \longrightarrow M_2$ homeomorphism



The topological equivalence h does not distinguish between the node and the focus, for example dicritical node can be by homeomorphism h mapped into the phase portrait of the stable focus.
In order to distinguish between the node and the focus, h must be a diffeomorphism, i.e., h must be continuous bijection (one-to-one and onto function) with a continuous inverse h^{-1} and partial derivatives of both h and h^{-1} must be also continuous.

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Remark Let systems (3) and (4) be topologically equivalent through the homeomorphism h. Then

(i) h maps stable (unstable) equilibria of the system (3) to the stable (unstable) equilibria of the system (4),

(ii) h maps closed trajectories into closed trajectories with the same period,

(iii) h maps $\omega-$limit sets of the trajectories of the system (3) into $\omega-$limit sets of the trajectories of the system (4),

(iv) h maps homoclinic (heteroclinic) trajectories of the system (3) to homoclinic (heteroclinic) trajectories of the system (4).

Definition Let the trajectory $\gamma_{\mathbf{x}}$ correspond to the solution $\varphi_{\mathbf{x}}(t)$ of the system $\dot{\mathbf{x}} = \mathbf{v}(\mathbf{x}(t))$. If there is a sequence $\{t_i\}_{i=1}^{\infty}$, $\lim\limits_{i\longrightarrow\infty} t_i = \infty$ such that $\lim\limits_{i\longrightarrow\infty} \varphi_{\mathbf{x}}(t_i) = \mathbf{z} \in \mathbb{R}^n$ exists, we call the point $\mathbf{z}$ an $\omega-$limit point of the trajectory $\gamma_{\mathbf{x}}$.
The set of all $\omega-$limit points = $\omega-$limit set of the trajectory $\gamma_{\mathbf{x}}$, denoted as $\omega(\gamma_{\mathbf{x}})$ or simply $\omega(\mathbf{x})$.

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If $\mathbf{x}_1$ is a steady state of the system $\dot{\mathbf{x}} = \mathbf{v}(\mathbf{x}(t))$, $\gamma_\mathbf{a}$ the trajectory of the solution $\varphi_\mathbf{a}(t)$, for which it holds

(i) $\lim_{t\to\infty} \varphi_\mathbf{a}(t) = \mathbf{x}_1$,

(ii) $\lim_{t\to\infty} \varphi'_\mathbf{a}(t) = \vec{\tau}_1$,

we say that the trajectory $\gamma_\mathbf{a}$ approaches the equilibrium state $\mathbf{x}_1$ in the direction of the vector $\vec{\tau}_1$.

Analogously, if for the equilibrium $\mathbf{x}_2$ it holds

(i) $\lim_{t\to-\infty} \varphi_\mathbf{b}(t) = \mathbf{x}_2$

(ii) $\lim_{t\to-\infty} \varphi'_\mathbf{b}(t) = \vec{\tau}_2$

we say that the trajectory $\gamma_\mathbf{b}$ of the solution $\varphi_\mathbf{b}(t)$ moves away from the equilibrium state $\mathbf{x}_2$ in the direction of the vector $\vec{\tau}_2$.

Remark If only the first relation is true and $\lim_{t\to\infty} \varphi'_\mathbf{a}(t)$ doesn't exist, we say that the trajectory ends in the point $\mathbf{x}_1$ (trajectory enters the equilibrium in spiral). Analogously, if $\lim_{t\to-\infty} \varphi'_\mathbf{b}(t)$ doesn't exist, the trajectory starts in the equilibrium $\mathbf{x}_2$.

★ Differentiable equivalence between systems

Let us have again two systems

$$\dot{\mathbf{x}} = \mathbf{v}(\mathbf{x}),\ \mathbf{x} \in M_1 \subseteq \mathbb{R}^2, \quad (5)$$

$$\dot{\mathbf{x}} = \mathbf{u}(\mathbf{x}),\ \mathbf{x} \in M_2 \subseteq \mathbb{R}^2, \quad (6)$$

where M_1, M_2 are domains in $\mathbb{R}^2$, $h : M_1 \to M_2$ homeomorphism that maps the trajectories of the first system to the trajectories of the second system preserving the orientatation, i.e., the phase portraits of these systems are topologically equivalent $\underbrace{h(\varphi(t,\mathbf{x})) = \psi(t, h(\mathbf{x}))}$.

rewrite as $\psi^t(h(\mathbf{x})) = h(\varphi^t(\mathbf{x}))$, where h is the diffeomorphism,

$$h(\mathbf{x}) = h(x_1, x_2) = (h_1(x_1,x_2), h_2(x_1,x_2)) \quad \text{and}$$

$$h'(\mathbf{x}) = \begin{pmatrix} \frac{\partial h_1(\mathbf{x})}{\partial x_1} & \frac{\partial h_1(\mathbf{x})}{\partial x_2} \\ \frac{\partial h_2(\mathbf{x})}{\partial x_1} & \frac{\partial h_2(\mathbf{x})}{\partial x_2} \end{pmatrix} = \frac{\partial \mathbf{h}(\mathbf{x})}{\partial \mathbf{x}}$$

is the derivative of the diffeomorfism h (Jacobi matrix of the mapping h) .

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Definition We say that the phase portraits of the systems (5) and (6) are differentiable equivalent if there exists a diffeomorphism $h : M_1 \to M_2$, which maps the trajectories of the first system to trajectories of the second system preserving the orientatation, i.e., it holds

$$\psi^t(h(\mathbf{x})) = h(\varphi^t(\mathbf{x})) .$$

Remark $\dot{\mathbf{x}} = \mathbf{v}(\mathbf{x}),\ \mathbf{x} \in M_1 \subseteq \mathbb{R}^2,\ \mathbf{x}^*$ equilibrium of this system, $\mathbf{y}^* = h(\mathbf{x}^*) \ldots$ equilibrium of the system $\dot{\mathbf{x}} = \mathbf{u}(\mathbf{x}), \mathbf{x} \in M_2 \subseteq \mathbb{R}^2$. Let

$\mathbf{J}(\mathbf{x}^*)$ ... the matrix of linearization of the first system in the steady state $\mathbf{x}^*$

$\mathbf{J}(\mathbf{y}^*)$ ... the matrix of linearization of the second system in the steady state $\mathbf{y}^* = h(\mathbf{x}^*)$

Then $\mathbf{J}(\mathbf{y}^*) = h'(\mathbf{x}^*) \cdot \mathbf{J}(\mathbf{x}^*) \cdot (h'(\mathbf{x}^*))^{-1}$,

matrices of the linearization at the steady states are similar , i.e., they have the same eigenvalues .

Conclusion The differentiable equivalence distinguishes node and focus.

Hartman-Grobman Theorem

Theorem (Hartman-Grobman) Let the system $\dot{\mathbf{x}} = \mathbf{v}(\mathbf{x}), \mathbf{x} \in \mathbb{R}^n$, have an isolated steady state $\mathbf{x}^*$ such that the corresponding matrix of linearization $\mathbf{J}(\mathbf{x}^*)$ has all eigenvalues with nonzero real parts.
Then there exists $\mathcal{O}(\mathbf{x}^*)$ such that in $\mathcal{O}(\mathbf{x}^*)$ the phase portrait of the system $\dot{\mathbf{x}} = \mathbf{v}(\mathbf{x})$ is topologically equivalent with the phase portrait of the linear system $\dot{\mathbf{x}} = \mathbf{J}(\mathbf{x}^*) \cdot \mathbf{x}$,, i.e., the phase flows of systems

$$\dot{\mathbf{x}} = \mathbf{v}(\mathbf{x}) \text{ (nonlinear)} \quad \text{and} \quad \dot{\mathbf{x}} = \mathbf{J}(\mathbf{x}^*) \cdot \mathbf{x} \text{ (linear)}$$

are topologically equivalent through a suitable homeomorphism.

Remark This theorem essentially states that the nonlinear system $\dot{\mathbf{x}} = \mathbf{v}(\mathbf{x})$ is locally homeomorphic to the linear system $\dot{\mathbf{x}} = \mathbf{J}(\mathbf{x}^*) \cdot \mathbf{x}$.
For $n = 2$ (planar systems) it may be proved that if the linearization matrix $\mathbf{J}(\mathbf{x}^*)$ has eigenvalues $\lambda_{1,2} = a \pm i\,b, a \cdot b \neq 0$, then the trajectories have in the neighborhood of $\mathbf{x}^*$ a spiral shape and ends in the steady state $\mathbf{x}^*$ (if $\mathbf{x}^*$ is stable), if $\mathbf{x}^*$ is unstable the trajectories have also the shape of spirals but in this case they move away from $\mathbf{x}^*$.

Bendixon's criterion

The Bendixon's criterion tells us that if we can show that an orbit with an initial condition in a region is contained in that region for all future time then there must be a closed orbit in the region.

Theorem (Bendixon's criterion) Let us have a planar system of differential equations

$$\dot{\mathbf{x}} = \mathbf{v}(\mathbf{x}), \quad \mathbf{x} = (x_1, x_2) \in \mathbb{R}^2, \qquad \text{i.e.,} \quad \begin{array}{rcl} \dot{x}_1 & = & v_1(x_1, x_2) \\ \dot{x}_2 & = & v_2(x_1, x_2)\,. \end{array}$$

If

$$\operatorname{div} \mathbf{v}(\mathbf{x}) = \frac{\partial v_1(\mathbf{x})}{\partial x_1} + \frac{\partial v_2(\mathbf{x})}{\partial x_2} \neq 0$$

on a simply connected domain $D \subset \mathbb{R}^2$ then the system $\dot{\mathbf{x}} = \mathbf{v}(\mathbf{x})$ does not have in D any closed trajectory $\gamma \subset D$.

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Example The system of differential equations in the plane:

$$\begin{aligned} x' &= -y + x(1 - x^2 - y^2), \\ y' &= x + y(1 - x^2 - y^2), \end{aligned} \quad \text{i.e.,} \quad \begin{aligned} v_1(x, y) &= -y + x(1 - x^2 - y^2)\,, \\ v_2(x, y) &= x + y(1 - x^2 - y^2)\,. \end{aligned}$$

$$\frac{\partial v_1}{\partial x} = 1 - 3x^2 - y^2\,, \quad \frac{\partial v_2}{\partial y} = 1 - x^2 - 3y^2\,.$$

$$\mathrm{div}\mathbf{v}(\mathbf{x}) = 2(1 - 2(x^2 + y^2)) = 0 \iff x^2 + y^2 = \frac{1}{2}\,.$$

$$\mathrm{div}\mathbf{v}(\mathbf{x}) > 0 \iff x^2 + y^2 < \frac{1}{2} \implies$$

By Bendixson's criterion, inside the circle there cannot be any whole closed trajectory.

Remark The outside of the circle is not a simply connected domain and the criterion can not be used. We know nothing about the existence of a closed trajectory outside the circle.

Recommended literature

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