Boundary value problem for ordinary differential equations

Methods for nonlinear boundary value problems (BVP) split into two main groups: diference method and shooting methods.

1 Difference methods

We consider a 2-point boundary value problem for one differential equation of the 2.nd order

$$y'' = f(x, y, y') \tag{1}$$

with linear boundary conditions

$$\alpha_0 \; y(a) + \beta_0 \; y'(a) = \gamma_0 \,, \tag{2}$$
$$\alpha_1 \; y(b) + \beta_1 \; y'(b) = \gamma_1 \,. \tag{3}$$

We divide the interval $[a, b]$ by an equidistant grid of points (nodes) $x_0 = a,\ x_1, \ldots, x_N = b$, $x_i = a + i\,h$, $h = (b-a)/N$. The values of the wanted solution $y(x)$ will be approximated by the values $y_i \sim y(x_i)$ in the nodes x_i. The differential equation (1) is replaced by the difference formula at x_i

$$\frac{y_{i-1} - 2y_i + y_{i+1}}{h^2} = f\left(x_i, y_i, \frac{y_{i+1} - y_{i-1}}{2h}\right) \,, \qquad i = 1, 2, \ldots, N-1. \tag{4}$$

The scheme of appearance of such an ordered system of nonlinear equations. We can solve this system of nonlinear equations by the Newton method. If the function $f(x, y)$ is linear then we can use methods for solving system of linear equations.

2 Shooting methods

The main idea of the shooting method is to choose the remaining information in one x value so that we can start the integration (to shoot) and to observe, how the boundary condition in the other x value is satisfied (how the target is hit).

We consider the system of differential equations of the first order

$$\begin{aligned} y_1' &= f(x, y_1, y_2), \\ y_2' &= g(x, y_1, y_2). \end{aligned} \tag{5}$$

and boundary conditions

$$\begin{aligned} \alpha_0 y_1(a) &+ \beta_0 y_2(a) = \gamma_0, \\ \alpha_1 y_1(b) &+ \beta_1 y_2(b) = \gamma_1. \end{aligned} \tag{6}$$

Thus it is a problem with separated boundary conditions. As this is a problem of order 1 in $x = a$ (and also in $x = b$) we must choose one condition in $x = a$ (or in $x = b$). Assuming $\beta_0 \neq 0$ we choose the initial condition

$$y_1(a) = \eta_1 \tag{7}$$

and we compute

$$y_2(a) = \eta_2 = \frac{\gamma_0 - \alpha_0 \eta_1}{\beta_0} \tag{8}$$

from the first equation (6). When integrating (5) with the initial conditions (7) and (8) we get $y_1(b) = y_1(b, \eta_1)$ and $y_2(b) = y_2(b, \eta_1)$, dependent on the choice of η_1. These values must satisfy the boundary conditions (6). The first of them is automatically satisfied by the choice of (8), the second one can be written as

$$\alpha_1 y_1(b, \eta_1) + \beta_1 y_2(b, \eta_1) - \gamma_1 = G(\eta_1) = 0 \ . \tag{9}$$

Now, after choosing η_1, we can compute the value of $G(\eta_1)$ according to (9) using some method for numerical integration of initial value problem. To solve the equation $G(\eta_1) = 0$ we use some method from Seminar 5 (Newton's method). Efficient methods use derivatives, an example being the Newton's method. The derivative can be found using some difference formula, but this is not very precise, since the numerical integration itself introduces certain error. A better choice is to consider variation

$$p_1 = \frac{\partial y_1}{\partial y_1(a)} = \frac{\partial y_1}{\partial \eta_1} \ , \qquad p_2 = \frac{\partial y_2}{\partial y_1(a)} = \frac{\partial y_2}{\partial \eta_1} \ . \tag{10}$$

The equations for p_1 and p_2 can be derived by differentiating (5) with respect to η_1 and interchanging the differentiation with respect to x and η_1

$$\begin{aligned} p_1' &= \frac{\partial f}{\partial y_1} p_1 + \frac{\partial f}{\partial y_2} p_2 \ , \\ p_2' &= \frac{\partial g}{\partial y_1} p_1 + \frac{\partial g}{\partial y_2} p_2 \end{aligned} \tag{11}$$

with the initial conditions

$$p_1(a) = 1 \ , \qquad p_2(a) = -\frac{\alpha_0}{\beta_0} \tag{12}$$

derived from (7) and (8). From (9) we have

$$\frac{\mathrm{d}G(\eta_1)}{\mathrm{d}\eta_1} = \alpha_1 p_1(b) + \beta_1 p_2(b) \ . \tag{13}$$

Then the Newton's method can be written as

$$\eta_1^{k+1} = \eta_1^k - \frac{G(\eta_1^k)}{G'(\eta_1^k)} = \eta_1^k - \frac{\alpha_1 y_1(b) + \beta_1 y_2(b) - \gamma_1}{\alpha_1 p_1(b) + \beta_1 p_2(b)} \ , \tag{14}$$

where $y_1(b), y_2(b), p_1(b), p_2(b)$ are evaluated for $\eta_1 = \eta_1^k$.

Now we consider instead of the system of differential equations (5) the differential equation of the second order

$$y'' = f(x, y, y'), \tag{15}$$

with boundary conditions

$$\begin{aligned} \alpha_0 y(a) + \beta_0 y'(a) &= \gamma_0, \\ \alpha_1 y(b) + \beta_1 y'(b) &= \gamma_1. \end{aligned} \tag{16}$$

This differential equations can be converted into a system of the first order

$$\begin{aligned} y_1' &= y_2, \\ y_2' &= f(x, y_1, y_2), \end{aligned} \tag{17}$$

with boundary conditions

$$\begin{aligned} \alpha_0 y_1(a) + \beta_0 y_2(a) &= \gamma_0, \\ \alpha_1 y_1(b) + \beta_1 y_2(b) &= \gamma_1. \end{aligned} \tag{18}$$

Now we can use the shooting method.