Numerical integration

We want to find the value of the definite integral

$$I(f) = \int_a^b f(x)\,dx\,.$$

To construct the approximation of the definite integral, we first define the mesh points x_i according to

$$a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b\,.$$

Typically the approximation will be of the form

$$I(f) \approx I_n(f) = \sum_{i=0}^{n} c_i\, f(x_i)\,.$$

1 Trapezoid Rule

We use a uniform grid, in which the mesh points are equally spaced so that $x_{i+1} - x_i = h$, for all $i = 0, \ldots, n-1$. We apply basic trapezoid rule

$$\int_{x_i}^{x_{i+1}} f(x)\,dx \approx \frac{h}{2}(f(x_{i+1}) + f(x_i))$$

to get the trapezoid rule

$$T_n(f) = \frac{h}{2}(f(x_0) + 2\sum_{i=1}^{n-1} f(x_i) + f(x_n))\,.$$

Trapezoid Rule Error Estimate
Let $f \in C^2([a,b])$ and let $T_n(f)$ be the trapezoid rule approximation to $I(f)$, using a uniform grid ($n+1$ points with $h = \frac{b-a}{n}$). Then there exists $\xi \in [a,b]$, depending on h, such that

$$I(f) - T_n(f) = -\frac{b-a}{12} h^2 f''(\xi)\,.$$

2 Simpson's Rule

Let $n = 2m$. We apply basic Simpson's rule

$$\int_{x_i}^{x_{i+2}} f(x)\,dx \approx \frac{h}{3}(f(x_{i-1}) + 4f(x_i) + f(x_{i+1}))$$

to get the Simpson's rule

$$S_n(f) = \frac{h}{3}(f(x_0) + 4\sum_{i=1}^{m} f(x_{2i-1}) + 2\sum_{i=1}^{m-1} f(x_{2i}) + f(x_{2m}))\,.$$

Simpson's Rule Error Estimate
Let $f \in C^4([a,b])$ and let $S_n(f)$ be the Simpson's rule approximation to $I(f)$, using a uniform grid ($n+1$ points with $h = \frac{b-a}{n}$). Then there exists $\xi \in [a,b]$, depending on h, such that

$$I(f) - S_n(f) = -\frac{b-a}{180} h^4 f^{(4)}(\xi)\,.$$

Exercises:
Write a computer program for the trapezoid rule and the Simpson's rule. Apply trapezoid and Simpson's rules with $h = 0.25$, 0.1, 0.05 to approximation the integral

1. $I = \int_0^1 x(1-x^2)\,dx = \frac{1}{4}$
2. $I = \int_0^1 \ln(1+x)\,dx = 2\ln 2 - 1$
3. $I = \int_0^1 \frac{1}{1+x^3}\,dx = \frac{1}{3}\ln 2 + \frac{1}{9}\sqrt{3}\pi$
4. $I = \int_1^2 \mathrm{e}^{-x^2}\,dx = 0.1352572580$

How small does the error theory say h must be to get the error less than 10^{-3} ? 10^{-6} ?