Numerical methods of the linear algebra

1 Linear systems and Gaussian elimination

We construct the algorithm to solve a linear system $\boldsymbol{Ax} = \boldsymbol{b}$ by Gaussian elimination. Here $\boldsymbol{A} = (a_{ij})_{i,j=1,\dots,n}$ is square matrix, $\boldsymbol{x} = (x_1, \dots, x_n)^T$ is the vector of unknowns and $\boldsymbol{b} = (b_1, \dots, b_n)^T$ is the right-side vector.
Suppose that $\boldsymbol{B} = [\boldsymbol{A}|\boldsymbol{b}]$. The Gaussian elimanation is to eliminate all the components of $\boldsymbol{B}$ below the the main diagonal, i.e. we reduce $\boldsymbol{B}$ to triangular form $\boldsymbol{B}'$. Suppose that $\boldsymbol{B}' = [\boldsymbol{A}'|\boldsymbol{b}']$. We are computing now

$$x_i = \frac{1}{a'_{ii}}(b'_i - \sum_{j=i+1}^{n} a'_{ij}x_j)\,.$$

This algorithm simply marches backward up the diagonal.

2 Perturbation, conditioning and stability

We will study how the matrix solution process is affected by small changes to the problem. The main reason for doing this is to understand of why some linear system problems are difficult to solve. Usually this involves a concept known as "conditioning".

2.1 Vector and Matrix Norms

Because we will measure errors in vectors and matrices, it is necessary to introduce the concept of vector norm and matrix norm. The most common examples of vector norms are

p-norm $||\boldsymbol{x}||_p = (\sum_{i=1}^{n} |x_i|^p)^{1/p}\ ,\quad p \geq 1$

1-norm $||\boldsymbol{x}||_1 = \sum_{i=1}^{n} |x_i|$

Euclidean 2-norm $||\boldsymbol{x}||_2 = \sqrt{\sum_{i=1}^{n} |x_i|^2}$

infinity norm $||\boldsymbol{x}||_\infty = \max_{1\leq i\leq n} |x_i|\,.$

Let $||.||$ be a given vector norm defined on $\mathbb{R}^n$. Define the corresponding matrix norm, for matrices $\boldsymbol{A} \in \mathbb{R}^{n\times n}$, by

$$||\boldsymbol{A}|| = \max_{\boldsymbol{x}\neq\boldsymbol{0}} \frac{||\boldsymbol{Ax}||}{||\boldsymbol{x}||}\,.$$

The matrix 1-norm $||\boldsymbol{A}||_1$ is defined as the maximum column sum:

$$||\boldsymbol{A}||_1 = \max_{1 \le j \le n} \sum_{i=1}^{n} |a_{ij}| \,.$$

The matrix infinity-norm $||\boldsymbol{A}||_\infty$ is defined as the maximum row sum:

$$||\boldsymbol{A}||_\infty = \max_{1 \le i \le n} \sum_{j=1}^{n} |a_{ij}| \,.$$

For a given matrix $\boldsymbol{A}$ and a given matrix norm $||.||$, the **condition number** with respect to the given norm is defined by

$$\kappa(\boldsymbol{A}) = ||\boldsymbol{A}|| \, ||\boldsymbol{A}^{-1}|| \,.$$

If $\boldsymbol{A}$ is singular, then we take $\kappa(\boldsymbol{A}) = \infty$.
The condition number measures how close the matrix is to being singular. One of the lessons of this is that systems that are nearly singular can produce large errors. If the matrix is ill conditioned (meaning that the condition number is large) then a small change in the data could lead to a large change in the solution.

2.2 Estimating the Condition Number

If we choose some vector $\boldsymbol{y} \in \mathbb{R}^n$, then the condition number estimate as

$$\kappa(\boldsymbol{A}) \ge ||\boldsymbol{A}|| \frac{||\boldsymbol{y}||}{||\boldsymbol{A}\boldsymbol{y}||}$$

Exercises:

1. Write a computer program for the Gaussian elimination.
2. Let
$$\boldsymbol{A} = \begin{pmatrix} 5 & 6 & -9 \\ 1 & 2 & 3 \\ 0 & 7 & 2 \end{pmatrix},$$
Compute $||\boldsymbol{A}||$
3. Let
$$\boldsymbol{A} = \begin{pmatrix} 2 & 2 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix},$$
This has exact condition number $\kappa(\boldsymbol{A}) = 8$. Use the condition number estimator to approximate the condition number.

4. Consider the linear system

$$\begin{pmatrix} 1.002 & 1 \\ 1 & 0.998 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0.002 \\ 0.002 \end{pmatrix},$$

which has the exact solution $(1, -1)^T$ (verify this). Make a "small" perturbation of the right-hand-side vector

$$\begin{pmatrix} 1.002 & 1 \\ 1 & 0.998 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0.003 \\ 0.002 \end{pmatrix}.$$

Find the solution and compare this solution with the previous one.