Newton's method

A fundamental problem of algebra is finding the root of the equation $\boldsymbol{f}(\boldsymbol{x}) = \mathbf{0}$. Newton's method is the classic algorithm for finding roots of functions.

1 Newton's method for the solution of a single nonlinear equation

Let $f : \mathbb{R} \to \mathbb{R}$, find the root $\bar{x}$ of this function ($f(\bar{x}) = 0$). We choose x_0 and we define x_{n+1} by

$$x_{n+1} = x_n - \frac{f(x_n}{f'(x_n)} .$$

Let $\bar{x}$ be the exact solution. Ideally, we would stop the Newton's method when the error $\bar{x} - x_n$ is sufficiently small. Unfortunately, we do not know the root $\bar{x}$ exactly. Usually we stop the iterations when $|x_n - x_{n+1}|$ is small, for example

$$|x_n - x_{n+1}| < \varepsilon ,$$

where $\varepsilon > 0$ is a user-defined tolerance.

2 Newton's method for the solution of a system of nonlinear equations

Now we will extend the Newton's method for a system of nonlinear equations $\boldsymbol{f}(\boldsymbol{x}) = \mathbf{0}$. Let $A_{\boldsymbol{f}}(\boldsymbol{x})$ be the Jacobi matrix of the map $\boldsymbol{f}$,

$$A_{\boldsymbol{f}}(\boldsymbol{x}) = \left(\frac{\partial f_i}{\partial x_j}(\boldsymbol{x}) \right)_{i,j=1,\ldots,n} .$$

We choose $\boldsymbol{x}_0$ and we define $\boldsymbol{x}_{n+1}$ by

$$\boldsymbol{x}_{n+1} = \boldsymbol{x}_n + \Delta \boldsymbol{x} ,$$

where $\Delta \boldsymbol{x}$ is the solution of this linear system of equations

$$A_{\boldsymbol{f}}(\boldsymbol{x_n}) \, \Delta \boldsymbol{x} = -\boldsymbol{f}(\boldsymbol{x}_n) .$$

Generalizing the notation, then, we have

$$\boldsymbol{x}_{n+1} = \boldsymbol{x}_n - A_{\boldsymbol{f}}^{-1}(\boldsymbol{x}_n) \, \boldsymbol{f}(\boldsymbol{x}_n) .$$

We stop the iterations when $||x_n - x_{n+1}|| < \varepsilon$, where $\varepsilon > 0$ is a user-defined tolerance.
Exercises:
Write a computer program that uses the Newton's method to find the root of the following function. Choose suitable x_0.

1. $f(x) = x^3 - 2x - 5$
2. $f(x) = x + \ln(x)$
3. $f(x) = x + \operatorname{tg} x$