Numerical methods for Ordinary Differential Equations

Let's consider the initial value problems for ordinary differential equations

$$y' = f(x, y)\,; \;\; y(x_0) = y_0\,,$$

where f is a known funkcion of x and y, and x_0, y_0 are given values.

1 Single–step methods: Runge–Kutta

The general Runge – Kutta method is in the form

$$\begin{aligned}
k_1 &= hf(x_n, y_n) \\
k_2 &= hf(x_n + \alpha_1 h, y_n + \beta_{11} k_1) \\
k_3 &= hf(x_n + \alpha_2 h, y_n + \beta_{21} k_1 + \beta_{22} k_2) \\
&\vdots \\
k_{j+1} &= hf(x_n + \alpha_j h, y_n + \beta_{j1} k_1 + \beta_{j2} k_2 + \ldots + \beta_{jj} k_j) \\
y_{n+1} &= y_n + \gamma_1 k_1 + \gamma_2 k_2 + \ldots + \gamma_{j+1} k_{j+1}\,.
\end{aligned}$$

where the $\alpha_1, \ldots, \alpha_j, \beta_{11}, \ldots, \beta_{jj}, \gamma_1, \ldots, \gamma_j$ are constants to be determined. For this we use Taylor's expansion.

1.1 Modified Euler's method

$$\begin{aligned}
k_1 &= h\,f(x_n, y_n) \\
k_2 &= h\,f(x_n + \frac{h}{2}, y_n + \frac{1}{2} k_1) \\
y_{n+1} &= y_n + k_2\,.
\end{aligned}$$

The global error of this method is $\mathcal{O}(h^2)$.

1.2 Standard fourth–order Runge–Kutta method

$$\begin{aligned}
k_1 &= h\,f(x_n, y_n) \\
k_2 &= h\,f(x_n + \frac{h}{2}, y_n + \frac{1}{2} k_1) \\
k_3 &= h\,f(x_n + \frac{h}{2}, y_n + \frac{1}{2} k_2) \\
k_4 &= h\,f(x_n + h, y_n + k_3) \\
y_{n+1} &= y_n + (k_1 + 2\,k_2 + 2\,k_3 + k_4)/6\,.
\end{aligned}$$

The global error of this method is $\mathcal{O}(h^4)$.

Exercises:

1. Write a computer program that solve the initial value problem using fourth-order Runge-Kutta method. Apply this program with $h = 1/10$ to find the solution of the following ODEs. Compare this results with the exact solutions.

   (a) $y' = 1 - 2\,y\,,\ y(0) = 1\,;\ y(x) = \frac{1}{2}e^{-2x}\,(1 + e^{2x})\,.$

   (b) $y' = (1 - y)\,y\,,\ y(0) = 1/2\,;\ y(x) = \frac{e^x}{1+e^x}.$

   (c) $y' = (1 + \mathrm{e}^{2x})\,y\,,\ y(0) = 1\,;\ y(x) = e^{x+\frac{e^{2x}}{2}-\frac{1}{2}}.$

   (d) $y' = -y + \sin(x),\ y(0) = 1/2\,;\ y(t) = \frac{1}{2}\,(-\cos(t) + 3e^{-t} + \sin(t))\,.$

2. For each initial problem below, approximate the solution using the standard fourth–order Runge–Kutta method with sequence of decreasing grids $h = \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$. For those problems when an exact solutions is given, compare the accuracy achieved over the interval $[0, 1]$ with theoretical accuracy.

   (a) $y' = 1 - 4\,y\,,\ y(0) = 1\,;\ y(x) = \frac{1}{4}(3\mathrm{e}^{-4x} + 1).$

   (b) $y' = -y\,\ln y\,,\ y(0) = 3\,;\ y(x) = \mathrm{e}^{(\ln 3)\mathrm{e}^{-x}}.$

   (c) $y' + y^2 = 0,\ y(0) = 1\,;\ y(x) = \frac{1}{x+1}.$