# Numerical methods for parabolic partial differential equations

Let's consider the linear parabolic equation with constant coefficients

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}\,. \tag{1}$$

A more general equation (describing heat conduction or mass diffusion)

$$\frac{\partial u}{\partial \tau} = \sigma \frac{\partial^2 u}{\partial x^2} \tag{2}$$

can be converted to (1) by the substitution $t = \sigma\tau$ .
The solution of equation (1) is often defined on a rectangle $D = [0,1]\times[0,T]$. The solution $u(x,t)$ satisfy the initial condition (the function $\varphi(x)$ is given)

$$u(x,0) = \varphi(x)\,, \qquad 0 < x < 1\,, \tag{3}$$

and a boundary condition

$$u(0,t) = u(1,t) = 0\,. \tag{4}$$

## 1 Simple explicit formula

Let us divide the interval $[0,1]$ in $x$ into $n$ subintervals by equidistant grid points

$$x_0 = 0,\ x_1 = h,\ x_2 = 2h,\ \ldots,\ x_{n-1} = 1-h,\ x_n = 1\,,$$

where $h = 1/n$ and $x_i = ih$, $i = 0,1,\ldots,n$. Similarly the interval $[0,T]$ in $t$ is divided into $r$ equal parts by the grid points

$$t_0 = 0,\ t_1 = k,\ \ldots,\ t_r = T,$$

where the time step is $k = T/r$ and $t_j = jk$, $j = 0,1,\ldots,r$. The set of nodes - the intersections of the lines $x = ih$, $i = 0,1,\ldots,n$, and the lines $t = jk$, $j = 0,1,\ldots,r$, forms a rectangular grid denoted by $D^{(h)}$. On this grid we can approximate the derivatives of the function $u$ by the difference formulas for $i = 1,\ldots,n-1,\ j = 0,\ldots,r-1$:

$$\left.\frac{\partial u}{\partial t}\right|_{(x_i,t_j)} = \frac{u_i^{j+1} - u_i^j}{k} + \mathcal{O}(k)\,, \tag{5}$$

$$\left.\frac{\partial^2 u}{\partial x^2}\right|_{(x_i,t_j)} = \frac{u_{i-1}^j - 2u_i^j + u_{i+1}^j}{h^2} + \mathcal{O}(h^2)\,, \tag{6}$$

where we denote $u(ih,jk) = u(x_i,t_j) \doteq u_i^j$.

Consider the equation (1) in one node $(x_i, t_j) \in D^{(h)}$ and the approximation using (5) and (6):

$$\frac{u_i^{j+1} - u_i^j}{k} = \frac{u_{i-1}^j - 2u_i^j + u_{i+1}^j}{h^2} + \mathcal{O}(k + h^2)\,. \tag{7}$$

Neglecting $\mathcal{O}(k+h^2) = \mathcal{O}(k) + \mathcal{O}(h^2)$, which is called the approximation error and using the initial condition (3) and the boundary conditions (4) we get the following difference problem:

$$u_i^{j+1} = \frac{k}{h^2}\left(u_{i-1}^j + u_{i+1}^j\right) + \left(1 - \frac{2k}{h^2}\right)u_i^j\,, \qquad \begin{array}{c} i = 1, 2, \ldots, n-1 \\ j = 0, 1, \ldots, r-1\,, \end{array} \tag{8}$$

$$u_i^0 = \varphi(ih)\,, \qquad i = 1, 2, \ldots, n-1\,, \tag{9}$$

$$u_0^j = 0\,, \qquad u_n^j = 0\,, \qquad j = 0, 1, \ldots, r\,. \tag{10}$$

It is easy to rewrite (8), (9) and (10) using profiles as

$$\begin{aligned} \boldsymbol{u}^{j+1} &= \mathbf{A}_1 \boldsymbol{u}^j\,, \\ \boldsymbol{u}^0 &= \left[0, \varphi(h), \varphi(2h), \ldots, \varphi((n-1)h), 0\right]^T\,, \end{aligned} \tag{11}$$

where the matrix $\mathbf{A}_1$ is three-diagonal

$$\mathbf{A}_1 = \begin{pmatrix} 0 & 0 & & & & & \\ \alpha & (1-2\alpha) & \alpha & & & & \\ & \alpha & (1-2\alpha) & \alpha & & 0 & \\ & & \ddots & \ddots & \ddots & & \\ & 0 & & \alpha & (1-2\alpha) & \alpha & \\ & & & & \alpha & (1-2\alpha) & \alpha \\ & & & & & 0 & 0 \end{pmatrix}\,, \tag{12}$$

where

$$\alpha = \frac{k}{h^2}\,. \tag{13}$$

If $\alpha \leq \frac{1}{2}$ then the method (11) is stable.