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ZIG-ZAG DYNAMICAL SYSTEMS AND THE BAKER-CAMPBELL-HAUSDORFF FORMULA

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ABSTRACT. Possibilities of using the Baker-Campbell-Hausdorff (BCH) formula to describe the ω-limit behavior of dynamical systems generated by two alternating vector fields (zig-zag dynamical systems) are studied. It is shown that in the case when the two vector fields generating the zig-zag dynamical system are linear the usage of the BCH formula is useful. Limitation for nonlinear case are discussed.

1. Introduction

In this paper we are going to study a particular class of dynamical systems that we chose to call "*zig-zag dynamical systems*" for its dynamics is determined by two smooth vector fields alternately operating on the phase space. Motivation for the study of such systems can be found in [5] and partially also in [6]. Both papers deal primarily with the study of periodic points of the zig-zag system, mostly in the case where the two vector fields are F-related by some involutive diffeomorphism F. In this article we will consider arbitrary smooth vector fields.

2. Preliminaries

Let M be a smooth manifold of dimension m with two smooth vector fields

$$\mathbf{u}, \mathbf{v} \colon M \to TM$$

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defined on it. The flows of these vector fields will be denoted by φ^t, ψ^t, where $t \in \mathbb{R}$, respectively.

Next, we consider a $2p$ periodic function $r_p(t)$ defined on $[0, 2p)$ by

$$r_p(t) = \begin{cases} 0 & \text{for } t \in [0, p)\,, \\ 1 & \text{for } t \in [p, 2p)\,. \end{cases} \tag{1}$$

The *zig-zag dynamical system* is now described by the following non-autonomous differential equation with a $2p$-periodic piece-wise continuous right hand side

$$\dot{x} = f(x,t) = \mathbf{u}(x) + r_p(t)\big[\mathbf{v}(x) - \mathbf{u}(x)\big]\,. \tag{2}$$

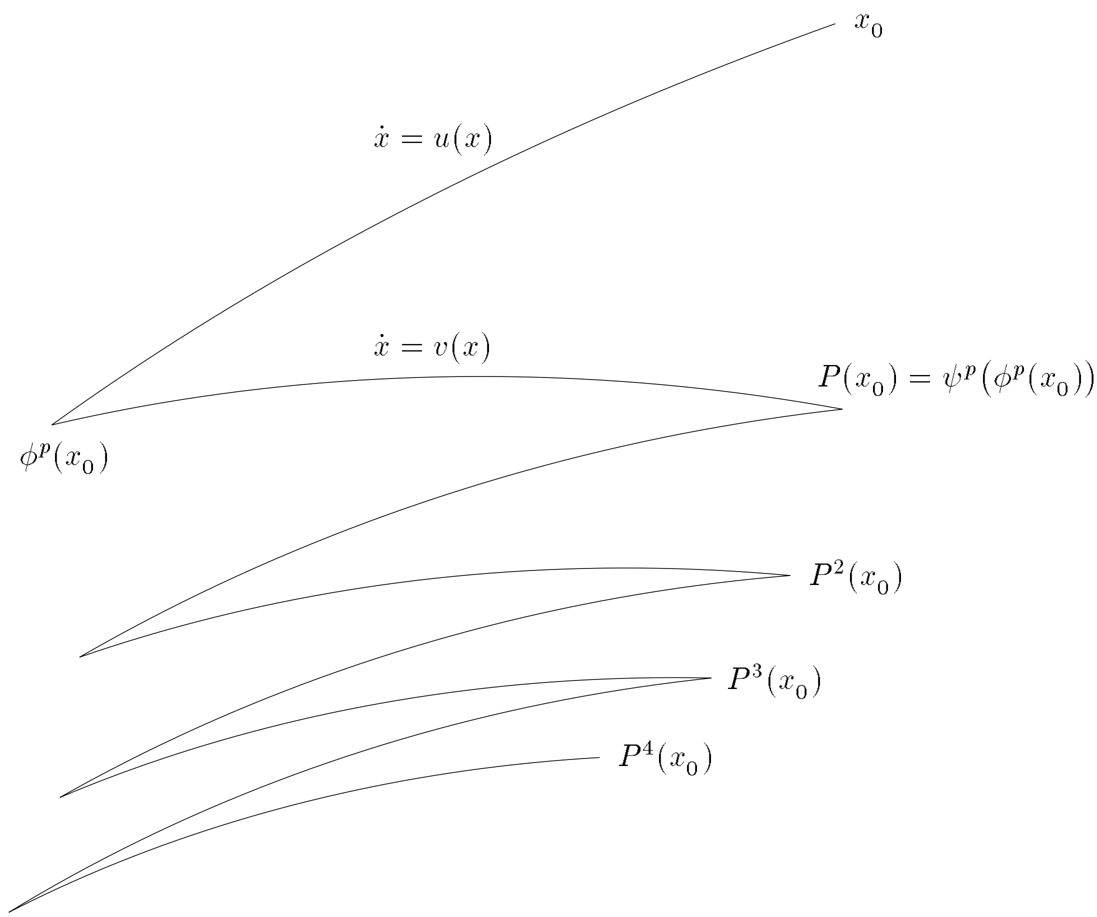

FIGURE 1. The trajectory of the zig-zag dynamical system. Starting from the initial condition x_0 the motion in the phase space is governed by the equation $\dot{x} = u(x)$ for the first half of the period and by the equation $\dot{x} = v(x)$ for the second half of the period. The vector field u has the flow φ^t and the vector field v has the flow ψ^t.

Let us denote by $\Phi(t;0,x_0)$ the solution of (2) satisfying the initial condition $\Phi(0;0,x_0)=x_0$. Using the phase flows φ^t and ψ^t we can express this solution for $t\in[0,2p)$ in the form

$$\Phi(t;0,x_0)=\begin{cases}\varphi^t(x_0) & \text{for } t\in[0,p)\,,\\ \psi^{t-p}(\varphi^p(x_0)) & \text{for } t\in[p,2p)\,.\end{cases} \tag{3}$$

Let us define

$$P(x)=\Phi(2p;0,x)\,. \tag{4}$$

Then the mapping P is a *period map* (also called a *stroboscopic map*) for the equation (2) and the ω-limit behavior of the solution of (2) can be described by the ω-limit behavior of the mapping P, i.e. by the ω-limit behavior of the orbit $\{x_0,P(x_0),P^2(x_0),\ldots\}=\{P^n(x_0)\}_{n=0}^{\infty}$.

From (3) and (4) it follows that

$$P=\psi^p\circ\varphi^p\,. \tag{5}$$

The situation is depicted on Fig. 1.

3. Averaging method and the BCH formula

In [5] we used the averaging method for finding periodic solutions of (2), with which we have associated an autonomous averaged system

$$\dot{x}=\mathbf{u}(x)+\mathbf{v}(x) \tag{6}$$

whose trajectories, as is well known, approximate solutions of (2) for small p on some finite interval. The right hand side of (6) is the first term of the Baker-Campbell-Hausdorff (BCH) formula, which, for the reader's convenience, we now briefly review, hoping that in the process we illustrate its importance in the study of the zig-zag systems.

The BCH formula for matrices.

From the classical theory of Lie groups and Lie algebras it follows that the matrix exponential satisfies

$$\mathrm{e}^{\mathbf{A}}\,\mathrm{e}^{\mathbf{B}}=\mathrm{e}^{(\mathbf{A}+\mathbf{B})}\qquad\text{if and only if}\quad \mathbf{AB}=\mathbf{BA}$$

i.e. if the two matrices commute. In case the matrices $\mathbf{A}$, $\mathbf{B}$ do not commute, then $\mathrm{e}^{\mathbf{A}}\,\mathrm{e}^{\mathbf{B}}=\mathrm{e}^{\mathbf{C}}$, where the matrix $\mathbf{C}$ is given by the BCH formula (see [10], [3])

$$\mathbf{C}=\mathbf{A}+\mathbf{B}+\frac{1}{2}[\mathbf{A},\mathbf{B}]+\frac{1}{12}\Big([\mathbf{A},[\mathbf{A},\mathbf{B}]]-[\mathbf{B},[\mathbf{A},\mathbf{B}]]\Big)+\cdots\,, \tag{7}$$

where $[\mathbf{A},\mathbf{B}]=\mathbf{AB}-\mathbf{BA}$ is the commutator of the matrices $\mathbf{A}$ and $\mathbf{B}$. We will use this fact in Sections 5 and 6.

The BCH formula for vector fields.

Here, we would like to provide some heuristic and motivational reasoning. Following [1], we denote by $\mathrm{Diff}_c^\infty(M)$ the group of C^∞ diffeomorphisms with compact support. The set $\mathrm{Diff}_c^\infty(M)$ is an infinite-dimensional Lie group, whose Lie algebra is the algebra $\mathcal{X}_c(M)$ of smooth vector fields on M. For $\mathbf{u}, \mathbf{v} \in \mathcal{X}_c(M)$ we denote by $[\mathbf{u}, \mathbf{v}]$ the usual Lie bracket of the two vector fields, (see e.g. [2]).

It is well known that a vector field $\mathbf{u} \in \mathcal{X}_c(M)$ generates a phase flow $\varphi^t \in \mathrm{Diff}_c^\infty(M)$, $t \in \mathbb{R}$. The diffeomorphism φ^1 is called the *time one map* of the flow φ^t. The correspondence

$$\mathbf{u} \mapsto \varphi^1$$

is a well defined mapping

$$\mathrm{Exp}\colon \mathcal{X}_c(M) \longrightarrow \mathrm{Diff}_c^\infty(M)\,, \tag{8}$$

that is called the *exponential mapping*. This mapping is an analogy of the exponential mapping for finite dimensional Lie groups.

Unfortunately, for the vector fields, this exponential mapping does not have the nice properties of the finite dimensional case (see [1], [4], [7], [11]), particularly it is neither one-to-one nor surjective near the identity (cf. [1], [7]), so there are diffeomorphisms from $\mathrm{Diff}_c^\infty(M)$ that are arbitrarily close to the identity, and yet do not have any pre-image in $\mathcal{X}_c(M)$ under the exponential mapping. The set

$$\mathrm{Exp}\big(\mathcal{X}_c(M)\big) = \mathcal{E} \tag{9}$$

will be called the *set of embeddable diffeomorphisms*, i.e. the diffeomorphisms that can be embedded into a flow φ^t of some vector field $\mathbf{u} \in \mathcal{X}_c(M)$. Typically, the set $\mathcal{E}$ is a rather irregular first category — like subset of $\mathrm{Diff}_c^\infty(M)$, (see [9]).

Let us now return to our zig-zag dynamical system and consider the vector fields $\mathbf{u}$, $\mathbf{v}$ from (2). Then

$$\mathrm{Exp}(\mathbf{u}) = \varphi^1$$

and

$$\mathrm{Exp}(\mathbf{v}) = \psi^1\,,$$

and thus

$$\varphi^p = \mathrm{Exp}(p\mathbf{u})$$

and

$$\psi^p = \mathrm{Exp}(p\mathbf{v})\,.$$

Let us assume, for this moment, the validity of the BCH formula even in this infinite-dimensional case, i.e. let us assume that the following holds

$$\big(\mathrm{Exp}(\mathbf{v})\big) \circ \big(\mathrm{Exp}(\mathbf{u})\big) = \mathrm{Exp}\big(\mathbf{w}(\mathbf{v} : \mathbf{u})\big)\,. \tag{10}$$

One might expect that then the vector field $\mathbf{w}(\mathbf{v} : \mathbf{u})$ is given by the BCH formula

$$\mathbf{w}(\mathbf{v} : \mathbf{u}) = (\mathbf{v} + \mathbf{u}) + \frac{1}{2}[\mathbf{v}, \mathbf{u}] + \frac{1}{12}\Big([\mathbf{v}, [\mathbf{v}, \mathbf{u}]] - [\mathbf{u}, [\mathbf{v}, \mathbf{u}]]\Big) + \cdots . \qquad (11)$$

In Paragraph 7 we show that this is not true and we derive a correct formula. With respect to relation (10) the period map P for the zig-zag system would be given by

$$P = \psi^p \circ \varphi^p = \operatorname{Exp}(p\mathbf{v}) \circ \operatorname{Exp}(p\mathbf{u}) = \operatorname{Exp}\big(\mathbf{w}(p\mathbf{v} : p\mathbf{u})\big) , \qquad (12)$$

where

$$\mathbf{w}(p\mathbf{v} : p\mathbf{u}) = p(\mathbf{v} + \mathbf{u}) + \frac{p^2}{2}[\mathbf{v}, \mathbf{u}] + \frac{p^3}{12}\Big([\mathbf{v}, [\mathbf{v}, \mathbf{u}]] - [\mathbf{u}, [\mathbf{v}, \mathbf{u}]]\Big) + \cdots . \qquad (13)$$

Then we could embed the period map P into the phase flow of the vector field $\mathbf{w}(p\mathbf{v} : p\mathbf{u})$. The orbits of the mapping P would lie on trajectories of the vector field $\mathbf{w}$ and the ω-limit behavior of these orbits will be determined by the ω-limit behavior of the trajectories of $\mathbf{w}$. At the same time we see that using the averaging method we obtain the first term in the BCH formula.

In Paragraph 4, the case is investigated, when the right-hand side of (11) is reduced to its first term and in Paragraphs 5 and 6 we will consider two cases in which the BCH formula for matrices can be applied for the study of qualitative properties of the linear zig-zag systems. Finally in Paragraph 7 the nonlinear case of zig-zag system is treated. In the last Paragraph 8 the numerical study is provided as a demonstration of using the BCH formula for an example of zig-zag system.

4. The case $[\mathbf{u}, \mathbf{v}] = 0$

In this paragraph, we will assume that the Lie bracket of the vector fields $\mathbf{u}, \mathbf{v}$ from (2) is equal to 0, i.e.

$$[\mathbf{u}, \mathbf{v}] = 0 . \qquad (14)$$

It is well known (see [2; p. 155, Theorem 7.12]) that the relation (14) is equivalent to

$$\varphi^t \circ \psi^s = \psi^s \circ \varphi^t \qquad \text{for all} \quad t, s \in \mathbb{R} , \qquad (15)$$

where φ^t and ψ^t are the phase flows of the vector fields $\mathbf{u}$, $\mathbf{v}$ respectively. In this case, the following theorem holds.

THEOREM 1. *Let* $\mathbf{u}$, $\mathbf{v}$ *be two vector fields in* $\mathcal{X}_c(M)$ *and* φ^t, ψ^t *their phase flows. If* $[\mathbf{u},\mathbf{v}]=0$, *then*

$$\eta^t = \psi^t \circ \varphi^t\,, \qquad t \in \mathbb{R}\,, \tag{16}$$

is the phase flow of the vector field $\mathbf{u}+\mathbf{v}$.

P r o o f. The first two properties of the flow are evident.
(i) $\eta^0 = \psi^0 \circ \varphi^0 = \mathrm{id}\,|_M$;
(ii) $\eta^t \circ \eta^s = \psi^t \circ \varphi^t \circ \psi^s \circ \varphi^s = \psi^{t+s} \circ \varphi^{t+s} = \eta^{t+s}$.
For (iii) we need to show that

$$\frac{\mathrm{d}\eta^t(x)}{\mathrm{d}t} = \mathbf{u}\big(\eta^t(x)\big) + \mathbf{v}\big(\eta^t(x)\big)\,.$$

We have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(\eta^t(x)) &= \frac{\mathrm{d}}{\mathrm{d}t}\big(\psi^t \circ \varphi^t(x)\big)\\ &= \frac{\partial\psi^t\big(\eta^t(x)\big)}{\partial t} + \frac{\partial\psi^t\big(\varphi^t(x)\big)}{\partial x}\frac{\mathrm{d}\varphi^t(x)}{\mathrm{d}t}\\ &= \mathbf{v}\big(\eta^t(x)\big) + \psi^t_*\big(\varphi^t(x)\big)\mathbf{u}\big(\varphi^t(x)\big)\,,\end{aligned}$$

or more briefly

$$\dot\eta^t = \dot\psi^t + \psi^t_*\dot\varphi^t = \mathbf{v} + \psi^t_*\mathbf{u}\,, \tag{17}$$

where we use the notation from [1; p. 3]. According to [2; p. 141, Theorem 5.7], we obtain that

$$\psi^t_*\mathbf{u} = \mathbf{u}\,,$$

so that

$$\dot\eta^t = \mathbf{u} + \mathbf{v}\,,$$

which was to be proved. □

Remark 4.1. In this case, the BCH formula takes the form

$$\mathrm{Exp}(\mathbf{v}) \circ \mathrm{Exp}(\mathbf{u}) = \mathrm{Exp}(\mathbf{u}+\mathbf{v}) \tag{18}$$

and the period map for the equation (2) is simply

$$P(x) = \eta^p(x)\,.$$

The orbits of the mapping P lie on the trajectories of the vector field $\mathbf{u}+\mathbf{v}$.

Remark 4.2. For each $\mathbf{u} \in \mathcal{X}_c(M)$ and for each $F \in \mathrm{Diff}^\infty_c(M)$

$$\mathrm{Exp}(F_*\mathbf{u}) = F \circ \mathrm{Exp}(\mathbf{u}) \circ F^{-1}\,, \tag{19}$$

as follows from considerations in [2; p. 137, Example 4]. The relationship (19) implies that for all $F \in \mathrm{Diff}^\infty_c(M)$

$$F \circ \big[\mathrm{Exp}\big(\mathcal{X}_c(M)\big)\big] \circ F^{-1} \subset \mathrm{Exp}\big(\mathcal{X}_c(M)\big)\,. \tag{20}$$

Let us now recall the Thurston's theorem (see [1; p. 24]):

Theorem T. *For any smooth manifold M, the identity component* $\mathrm{Diff}_c^\infty(M)_0$ *in* $\mathrm{Diff}_c^\infty(M)$ *is a simple group.*

This theorem and the relation (20) clearly imply that the set $\mathrm{Exp}\big(\mathcal{X}_c(M)\big)$ cannot be a subgroup of the group $\mathrm{Diff}_c^\infty(M)$ for if it were one, then by (20) it would have to be a normal subgroup which contradicts the Theorem T.

Remark 4.3. Suppose that

$$\mathrm{Exp}(\mathbf{u}) \circ \mathrm{Exp}(\mathbf{v}) = \mathrm{Exp}\big(\mathbf{w}(\mathbf{u} : \mathbf{v})\big)\,, \tag{21}$$

where $\mathbf{w} \in \mathcal{X}_c(M)$ i.e. the diffeomorphism $\varphi^1 \circ \psi^1$ is embeddable. Then for each $F \in \mathrm{Diff}_c^\infty(M)$ we get

$$\mathrm{Exp}(F_*\mathbf{u}) \circ \mathrm{Exp}(F_*\mathbf{v}) = \mathrm{Exp}\big(F_*\mathbf{w}(\mathbf{u} : \mathbf{v})\big)\,, \tag{22}$$

since on the left hand side we have, according to (19)

$$\begin{aligned} &F \circ \mathrm{Exp}(\mathbf{u}) \circ F^{-1} \circ F \circ \mathrm{Exp}(\mathbf{v}) \circ F^{-1} \\ &= F \circ \mathrm{Exp}(\mathbf{u}) \circ \mathrm{Exp}(\mathbf{v}) \circ F^{-1} \\ &= F \circ \mathrm{Exp}\big(\mathbf{w}(\mathbf{u} : \mathbf{v})\big) \circ F^{-1} \\ &= \mathrm{Exp}\big(F_*\mathbf{w}(\mathbf{u} : \mathbf{v})\big)\,. \end{aligned}$$

5. The linear zig-zag system

In this paragraph we set $M = \mathbb{R}^n$ and consider two linear vector fields

$$\mathbf{u}(\boldsymbol{x}) = \mathbf{A}\boldsymbol{x} \qquad \text{and} \qquad \mathbf{v}(\boldsymbol{x}) = \mathbf{B}\boldsymbol{x}\,, \tag{23}$$

where $\mathbf{A}$, $\mathbf{B}$ are arbitrary square matrices of order n, $\boldsymbol{x} \in \mathbb{R}^n$. The corresponding phase flows are then

$$\varphi^t(\boldsymbol{x}) = \mathrm{e}^{t\mathbf{A}}\,\boldsymbol{x} \qquad \text{and} \qquad \psi^t(\boldsymbol{x}) = \mathrm{e}^{t\mathbf{B}}\,\boldsymbol{x}\,, \tag{24}$$

so that

$$\mathrm{Exp}(\mathbf{A}\boldsymbol{x}) = \mathrm{e}^{\mathbf{A}}\,\boldsymbol{x} \qquad \text{and} \qquad \mathrm{Exp}(\mathbf{B}\boldsymbol{x}) = \mathrm{e}^{\mathbf{B}}\,\boldsymbol{x}\,.$$

That implies

$$\mathrm{Exp}(\mathbf{A}\boldsymbol{x}) \circ \mathrm{Exp}(\mathbf{B}\boldsymbol{x}) = \mathrm{e}^{\mathbf{A}}\,\mathrm{e}^{\mathbf{B}}\,\boldsymbol{x} = \mathrm{e}^{\mathbf{C}}\,\boldsymbol{x} = \mathrm{Exp}(\mathbf{C}\boldsymbol{x})\,, \tag{25}$$

where the matrix $\mathbf{C}$ is determined by (7).

Let us now return to the period map P for the equation (2), where the vector fields $\mathbf{u}$ and $\mathbf{v}$ are given by (23). Then, considering (24) we obtain

$$P(\boldsymbol{x}) = \psi^p \circ \varphi^p(\boldsymbol{x}) = \mathrm{e}^{p\mathbf{B}}\,\mathrm{e}^{p\mathbf{A}}\,\boldsymbol{x} = \mathrm{e}^{\mathbf{D}}\,\boldsymbol{x}\,,$$

where the matrix $\mathbf{D}$ is, according to (7), given by

$$\mathbf{D} = p(\mathbf{B}+\mathbf{A}) + \frac{p^2}{2}[\mathbf{B},\mathbf{A}] + \frac{p^3}{12}\Big([\mathbf{B},[\mathbf{B},\mathbf{A}]] - [\mathbf{A},[\mathbf{B},\mathbf{A}]]\Big) + \cdots . \tag{26}$$

Thus the mapping $P(\boldsymbol{x})$ can be embedded into the flow $\eta^t(\boldsymbol{x})$, which is the phase flow of the linear system

$$\dot{\boldsymbol{x}} = \mathbf{D}\boldsymbol{x}\,.$$

The orbits of the period map P are lying on the trajectories of this linear system. Thus we have proved the following theorem.

Theorem 2. *Let $\mathbf{A}$, $\mathbf{B}$ be matrices of order n and $r_p(t)$ be a function defined by (1). Let furthermore $P\colon \mathbb{R}^n \to \mathbb{R}^n$ be the period (time $2p$) map of the system*

$$\dot{\boldsymbol{x}} = \mathbf{A}\boldsymbol{x} + r_p(t)[\mathbf{B}\boldsymbol{x} - \mathbf{A}\boldsymbol{x}]\,.$$

Then for sufficiently small $p > 0$ there exists a matrix $\mathbf{D}$, given by (26), such that every orbit $O_P = \{P^n(\boldsymbol{x}_0) :\ n \in \mathbb{N}\}$ of the mapping P, lies on the trajectory of the linear system $\dot{\boldsymbol{x}} = \mathbf{D}\boldsymbol{x}$, passing though the initial state $\boldsymbol{x}_0$.

Remark 5.1. Let us return to the relation (25). It was obtained by using the BCH formula for matrices. We will show how to write this relation using the BCH formula for linear vector fields. For this purpose we must consider that the usual bracket product of linear vector fields satisfies

$$[\mathbf{A}\boldsymbol{x},\mathbf{B}\boldsymbol{x}] = \mathbf{B}\mathbf{A}\boldsymbol{x} - \mathbf{A}\mathbf{B}\boldsymbol{x} = -[\mathbf{A},\mathbf{B}]\boldsymbol{x}\,.$$

When we introduce a more convenient bracket product (see [7; p. 1041])

$$[\mathbf{v},\mathbf{u}]^{\sim} = -[\mathbf{v},\mathbf{u}]\,,$$

i.e. the appropriate bracket product is just the *negative* of the usual bracket product of vector fields, then

$$[\mathbf{A}\boldsymbol{x},\mathbf{B}\boldsymbol{x}]^{\sim} = [\mathbf{A},\mathbf{B}]\boldsymbol{x}$$

and the vector field $\mathbf{C}\boldsymbol{x}$ in (25) can be expressed in the form

$$\mathbf{C}\boldsymbol{x} = \mathbf{A}\boldsymbol{x} + \mathbf{B}\boldsymbol{x} + \frac{1}{2}[\mathbf{A}\boldsymbol{x},\mathbf{B}\boldsymbol{x}]^{\sim} + \frac{1}{12}\Big([\mathbf{A}\boldsymbol{x},[\mathbf{A}\boldsymbol{x},\mathbf{B}\boldsymbol{x}]^{\sim}]^{\sim} - [\mathbf{B}\boldsymbol{x},[\mathbf{A}\boldsymbol{x},\mathbf{B}\boldsymbol{x}]^{\sim}]^{\sim}\Big) + \cdots , \tag{27}$$

which is just the BCH formula for linear vector fields.

6. Nonhomogeneous linear zig-zag system

In this paragraph we set $M = \mathbb{R}^n$ again and consider two vector fields

$$\mathbf{u}(\boldsymbol{x}) = \mathbf{A}(\boldsymbol{x} - \boldsymbol{x}_0) \qquad \text{and} \qquad \mathbf{v}(\boldsymbol{x}) = \mathbf{B}(\boldsymbol{x} - \boldsymbol{x}_1)\,, \tag{28}$$

where $\boldsymbol{x}_0, \boldsymbol{x}_1 \in \mathbb{R}^n$ and $\boldsymbol{x}_0 \neq \boldsymbol{x}_1$.

Remark 6.1. The case when $\boldsymbol{x}_0 = \boldsymbol{x}_1$ can be converted to the linear case using (22) from the Remark 4.3. If we choose F in the form

$$F(\boldsymbol{x}) = \boldsymbol{x} + \boldsymbol{x}_0\,,$$

then $F_*(\mathbf{A}\boldsymbol{x}) = \mathbf{A}(\boldsymbol{x} - \boldsymbol{x}_0)$ and $F_*(\mathbf{B}\boldsymbol{x}) = \mathbf{B}(\boldsymbol{x} - \boldsymbol{x}_0)$ and we obtain easily a result similar to Theorem 2.

The vector fields (28) determine nonhomogeneous linear differential equations

$$\dot{\boldsymbol{x}} = \mathbf{A}(\boldsymbol{x} - \boldsymbol{x}_0) \qquad \text{and} \qquad \dot{\boldsymbol{x}} = \mathbf{B}(\boldsymbol{x} - \boldsymbol{x}_1)$$

with the corresponding phase flows

$$\varphi^t(\boldsymbol{x}) = \boldsymbol{x}_0 + \mathrm{e}^{t\mathbf{A}}(\boldsymbol{x} - \boldsymbol{x}_0) \qquad \text{and} \qquad \psi^t(\boldsymbol{x}) = \boldsymbol{x}_1 + \mathrm{e}^{t\mathbf{B}}(\boldsymbol{x} - \boldsymbol{x}_1)\,. \tag{29}$$

The corresponding period map has the form

$$P(\boldsymbol{x}) = \psi^p \circ \varphi^p(\boldsymbol{x}) = \boldsymbol{x}_1 + \mathrm{e}^{p\mathbf{B}}\big(\boldsymbol{x}_0 + \mathrm{e}^{p\mathbf{A}}(\boldsymbol{x} - \boldsymbol{x}_0) - \boldsymbol{x}_1\big)\,. \tag{30}$$

Let us set

$$\mathrm{e}^{p\mathbf{B}} \cdot \mathrm{e}^{p\mathbf{A}} = \mathrm{e}^{\mathbf{D}}\,,$$

where the matrix $\mathbf{D}$ is given by (26) and let us consider the differential equation

$$\dot{\boldsymbol{x}} = \mathbf{D}(\boldsymbol{x} - \boldsymbol{x}^*)$$

with the phase flow

$$\eta^t(\boldsymbol{x}) = \boldsymbol{x}^* + \mathrm{e}^{t\mathbf{D}}(\boldsymbol{x} - \boldsymbol{x}^*) \tag{31}$$

into which we want to embed the mapping $P = \psi^p \circ \varphi^p$ so that

$$\psi^p \circ \varphi^p = \eta^1\,. \tag{32}$$

Substituting (30) and (31) into (32) we get

$$\boldsymbol{x}_1 + \mathrm{e}^{p\mathbf{B}}\big[\boldsymbol{x}_0 + \mathrm{e}^{p\mathbf{A}}(\boldsymbol{x} - \boldsymbol{x}_0) - \boldsymbol{x}_1\big] = \boldsymbol{x}^* + \mathrm{e}^{\mathbf{D}}(\boldsymbol{x} - \boldsymbol{x}^*)$$

and after simplification

$$\big[\mathbf{E} - \mathrm{e}^{p\mathbf{B}}\,\mathrm{e}^{p\mathbf{A}}\big]\boldsymbol{x}^* = \boldsymbol{x}_1 + \mathrm{e}^{p\mathbf{B}}(\boldsymbol{x}_0 - \boldsymbol{x}_1) - \mathrm{e}^{p\mathbf{B}}\,\mathrm{e}^{p\mathbf{A}}\,\boldsymbol{x}_0\,, \tag{33}$$

where $\mathbf{E}$ is the identity matrix. Assuming that $\mathbf{E} - e^{\mathbf{D}}$ is regular, we can solve this equation for $\mathbf{x}^*$:

$$\mathbf{x}^* = \left[\mathbf{E} - e^{\mathbf{D}}\right]^{-1} \cdot \left[\mathbf{x}^1 + e^{p\mathbf{B}}(\mathbf{x}_0 - \mathbf{x}_1) - e^{\mathbf{D}}\,\mathbf{x}_0\right]. \tag{34}$$

This fully determines the phase flow (31).

Remark 6.2. The previous argument shows that the embedding of the diffeomorphism (30) into the flow (31) may not be always possible but if the matrix $\mathbf{E} - e^{\mathbf{D}}$ is not regular then the diffeomorphism (30) can still be embedded into a flow of a more general vector field, namely

$$\dot{\mathbf{x}} = \mathbf{D}\mathbf{x} + \mathbf{x}_{\mathbf{D}},$$

where $\mathbf{x}_{\mathbf{D}}$ is a constant vector.

EXAMPLE 6.1. As an illustration, we will discuss a simple example from [5], which is called "blinking nodes" there.

We will consider two two-dimensional vector fields

$$\mathbf{u}(x,y) = (-x+1, -y), \qquad \mathbf{v}(x,y) = (-x-1, -y), \tag{35}$$

and the corresponding differential equations

$$\dot{x} = -x + 1, \quad \dot{y} = -y,$$

and

$$\dot{x} = -x - 1, \quad \dot{y} = -y,$$

whose phase flows are described explicitly as

$$\varphi^t(x,y) = \left(1 + (x-1)\,e^{-t},\; y\,e^{-t}\right),$$
$$\psi^t(x,y) = \left(-1 + (x+1)\,e^{-t},\; y\,e^{-t}\right).$$

In this case, considering the notation (28), we obtain

$$\mathbf{A} = \mathbf{B} = -\mathbf{E}, \qquad \mathbf{x}_0 = (1,0) = -\mathbf{x}_1, \qquad p\mathbf{A} = p\mathbf{B} = -p\mathbf{E}.$$

The equation (34) has then the form

$$\begin{aligned}\mathbf{x}^* &= \left[\mathbf{E} - e^{-2p\mathbf{E}}\right]^{-1} \cdot \left[-\mathbf{x}_0 + 2\mathbf{x}_0\,e^{-p\mathbf{E}} - \mathbf{x}_0\,e^{-2p\mathbf{E}}\right] \\ &= \left[\mathbf{E} - e^{-2p\mathbf{E}}\right]^{-1} \cdot \left[-\mathbf{E} + 2\,e^{-p\mathbf{E}} - e^{-2p\mathbf{E}}\right]\mathbf{x}_0.\end{aligned}$$

Considering the matrix identities

$$\mathrm{e}^{-2p\mathbf{E}} = \begin{bmatrix} \mathrm{e}^{-2p} & 0 \\ 0 & \mathrm{e}^{-2p} \end{bmatrix},$$

$$\left[\mathbf{E} - \mathrm{e}^{-2p\mathbf{E}}\right]^{-1} = \begin{bmatrix} \frac{1}{1-\mathrm{e}^{-2p}} & 0 \\ 0 & \frac{1}{1-\mathrm{e}^{-2p}} \end{bmatrix}$$

and

$$\left(2\,\mathrm{e}^{-p\mathbf{E}} - \mathrm{e}^{-2p\mathbf{E}} - \mathbf{E}\right) = \begin{bmatrix} -(\mathrm{e}^{-p}-1)^2 & 0 \\ 0 & -(\mathrm{e}^{-p}-1)^2 \end{bmatrix},$$

we obtain

$$\boldsymbol{x}^* = \begin{bmatrix} \frac{(\mathrm{e}^{-p}-1)^2}{1-\mathrm{e}^{-2p}} & 0 \\ 0 & \frac{(\mathrm{e}^{-p}-1)}{1-\mathrm{e}^{-2p}} \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{\mathrm{e}^{-p}-1}{\mathrm{e}^{-p}+1} \\ 0 \end{bmatrix}.$$

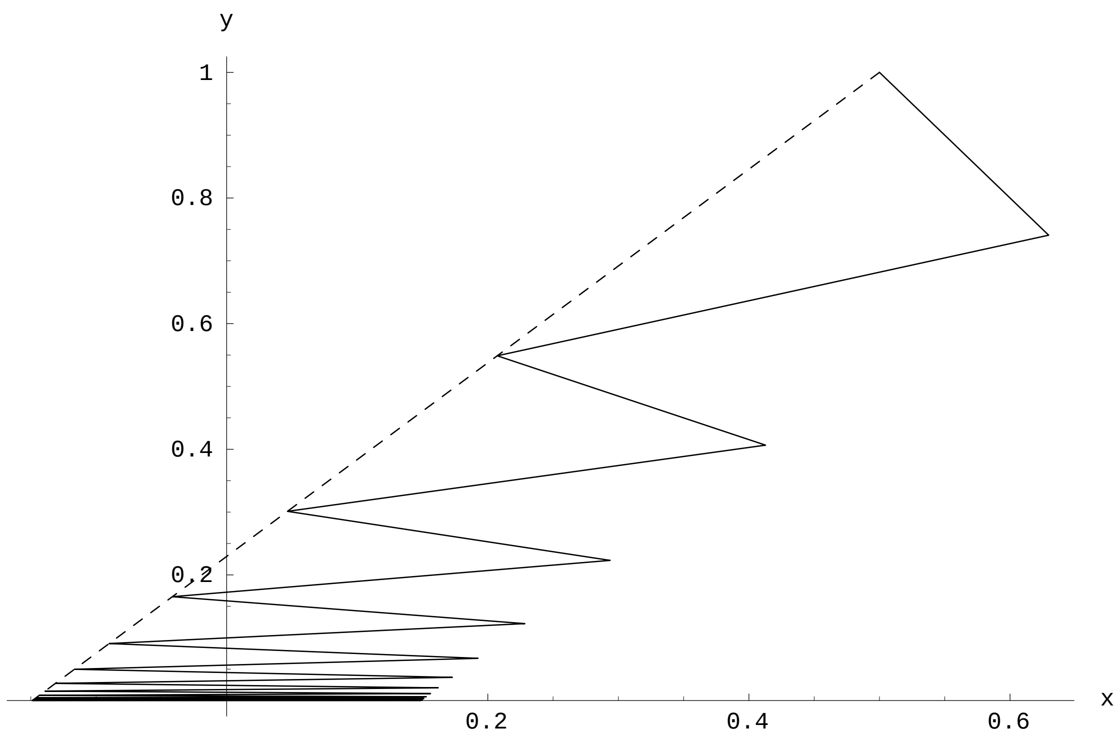

FIGURE 2. The solid line is the trajectory of the zig-zag system “blinking nodes”. Our goal is to find a vector field whose trajectory (the dashed line) contains the orbit of the period map of the “zig-zag” system. The point x^* is the attractor of the period map P.

The situation is depicted in Fig. 2, where the solution curve of the zig-zag system (2) is marked by a solid line, while the trajectory of the phase flow (31) is marked by a dashed line. We note that this result agrees with the result from [5], which was obtained by a different way.

7. Nonlinear zig-zag system

From the reasons discussed in [7] it follows that the generalization of the BCH formula from finite dimensional Lie groups to infinite dimensional Lie groups is possible only in the case when this infinite dimensional Lie group can be provided with a real analytic structure. This is not possible for $\mathrm{Diff}_c^\infty(M)$ (cf. [7]). Thus the BCH formula in general setting for the group $\mathrm{Diff}_c^\infty(M)$ does not even make sense.

Another difficulty with the BCH formula for the group $\mathrm{Diff}_c^\infty(M)$ follows from the Remark 4.2, namely even when the diffeomorphisms

$$\varphi^1 = \mathrm{Exp}(\mathbf{u}) \qquad \text{and} \qquad \psi^1 = \mathrm{Exp}(\mathbf{v})$$

are embeddable, then their composition $\psi^1 \circ \varphi^1$ is not necessarily embeddable, because the set $\mathrm{Exp}\big(\mathcal{X}_c(M)\big)$ is not a group.

Despite of this it seems to us reasonable to consider the following problem.

Problem Formulation. Let us consider two vector fields $\mathbf{u}, \mathbf{v} \in \mathcal{X}_c(M)$ (sufficiently "small") with the phase flows φ^t, ψ^t, i.e.

$$\varphi^1 = \mathrm{Exp}(\mathbf{u}) \qquad \text{and} \qquad \psi^1 = \mathrm{Exp}(\mathbf{v})\,.$$

Let us suppose that

$$\psi^1 \circ \varphi^1 \in \mathrm{Exp}\big(\mathcal{X}_c(M)\big)\,, \tag{36}$$

i.e. the composition of corresponding 1-flows is an embeddable diffeomorphism. That means there exists a vector field $\mathbf{w} \in \mathcal{X}_c(M)$ with the phase flow η^t satisfying

$$\eta^1 = \psi^1 \circ \varphi^1\,, \tag{37}$$

or

$$\mathrm{Exp}(\mathbf{w}) = \psi^1 \circ \varphi^1\,. \tag{38}$$

Let us try to find the vector field $\mathbf{w}$ satisfying (38). There may exist more than one such vector fields because the mapping Exp is not one-to-one. It is obvious that the vector field $\mathbf{w}$ depends on the two vector fields $\mathbf{u}$ and $\mathbf{v}$. We denote this dependence $\mathbf{w}(\mathbf{v} : \mathbf{u})$. We want to express the vector field $\mathbf{w}$ in terms of the two vector fields $\mathbf{v}$ and $\mathbf{u}$.

The method used below is based on the expression of the phase flow φ^t of a vector field $\mathbf{u} \in \mathcal{X}_c(M)$ using the relation [8; (1.19)], i.e. for $t \in \mathbb{R}$, $\boldsymbol{x} \in M$ we have the Lie series

$$\varphi^t(\boldsymbol{x}) = \sum_{k=0}^{\infty} \frac{t^k}{k!}\mathbf{u}^k(\boldsymbol{x}) = \boldsymbol{x} + t\mathbf{u}(\boldsymbol{x}) + \frac{t^2}{2}\mathbf{u}^2(\boldsymbol{x}) + \cdots\,. \tag{39}$$

For details see [8].

We will work with local coordinates and let us set $M = \mathbb{R}^n$ for simplicity and the vector fields will be given using the coordinate functions in the form

$$\mathbf{u}(\boldsymbol{x}) = \sum_{i=1}^{n} u_i(\boldsymbol{x}) \frac{\partial}{\partial x_i}\,,$$

$$\mathbf{v}(\boldsymbol{x}) = \sum_{i=1}^{n} v_i(\boldsymbol{x}) \frac{\partial}{\partial x_i}\,,$$

where $\boldsymbol{x} = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$. Let us recall that the Lie bracket of these vector fields is given by

$$[\mathbf{u}, \mathbf{v}] = \sum_{i=1}^{n} \left(\sum_{j=1}^{n} \left(u_j \frac{\partial v_i}{\partial x_j} - v_j \frac{\partial u_i}{\partial x_j} \right) \right) \frac{\partial}{\partial x_i}\,.$$

We will use the short form

$$[\mathbf{u}, \mathbf{v}] = \mathbf{u}(\mathbf{v}) - \mathbf{v}(\mathbf{u}) = \mathbf{v}'\mathbf{u} - \mathbf{u}'\mathbf{v}\,,$$

where

$$\mathbf{u}' = \left(\frac{\partial u_i}{\partial x_j} \right)_{i,j=1,\ldots,n}.$$

Further we will write briefly

$$\mathbf{u}^2 = \mathbf{u}(\mathbf{u}), \qquad \mathbf{u}^3 = \mathbf{u}(\mathbf{u}^2) \qquad \text{etc.}$$

Let us denote for $t = p$

$$\boldsymbol{y} = \varphi^p(\boldsymbol{x}) = \boldsymbol{x} + p\mathbf{u}(\boldsymbol{x}) + \frac{p^2}{2}\mathbf{u}^2(\boldsymbol{x}) + \cdots \tag{40}$$

and

$$\boldsymbol{z} = \psi^p(\boldsymbol{y}) = \boldsymbol{y} + p\mathbf{v}(\boldsymbol{y}) + \frac{p^2}{2}\mathbf{v}^2(\boldsymbol{y}) + \cdots\,. \tag{41}$$

Now we want to find a vector field $\mathbf{w}$ with the phase flow η^t such that

$$\eta^1 = \psi^p \circ \varphi^p\,, \tag{42}$$

i.e.

$$\eta^1(\boldsymbol{x}) = \boldsymbol{z} = \psi^p(\boldsymbol{y})\,. \tag{43}$$

We want to find the vector field $\mathbf{w} = \mathbf{w}(\mathbf{v} : \mathbf{u})$ as a formal series

$$\mathbf{w}(\boldsymbol{x}) = \mathbf{w}_1(\boldsymbol{x}) + \mathbf{w}_2(\boldsymbol{x}) + \mathbf{w}_3(\boldsymbol{x}) + \cdots = \sum_{k=1}^{\infty} \mathbf{w}_k(\boldsymbol{x})\,, \tag{44}$$

where $\mathbf{w}_k(\boldsymbol{x}) \in \mathcal{X}_c(M)$ and

$$\mathbf{w}_k(p\mathbf{v} : p\mathbf{u}) = p^k \mathbf{w}_k(\mathbf{v} : \mathbf{u}) . \tag{45}$$

By substituting from (40) into (41) we get after simplification

$$\begin{aligned}
\boldsymbol{z} = \boldsymbol{x} &+ p\mathbf{u}(\boldsymbol{x}) + \frac{p^2}{2}\mathbf{u}^2(\boldsymbol{x}) + \frac{p^3}{3!}\mathbf{u}^3(\boldsymbol{x}) + O(p^4) \\
&+ p\Big\{\mathbf{v}\Big(\boldsymbol{x} + p\mathbf{u}(\boldsymbol{x}) + \frac{p^2}{2}\mathbf{u}^2(\boldsymbol{x}) + O(p^3)\Big)\Big\} \\
&+ \frac{p^2}{2}\big\{\mathbf{v}^2\big(\boldsymbol{x} + p\mathbf{u}(\boldsymbol{x}) + O(p^2)\big)\big\} + \frac{p^3}{3!}\big\{\mathbf{v}^3\big(\boldsymbol{x} + O(p)\big)\big\} \\
= \boldsymbol{x} &+ p\mathbf{u}(\boldsymbol{x}) + \frac{p^2}{2}\mathbf{u}^2(\boldsymbol{x}) + \frac{p^3}{3!}\mathbf{u}^3(\boldsymbol{x}) + O(p^4) \\
&+ p\Big\{\mathbf{v}(\boldsymbol{x}) + \mathbf{v}'(\boldsymbol{x})\Big[p\mathbf{u}(\boldsymbol{x}) + \frac{p^2}{2}\mathbf{u}^2(\boldsymbol{x}) + O(p^3)\Big]\Big\} \\
&+ \frac{p^2}{2}\big\{\mathbf{v}^2(\boldsymbol{x}) + \big(\mathbf{v}^2(\boldsymbol{x})\big)'\big[p\mathbf{u}(\boldsymbol{x}) + O(p^2)\big]\big\} + \frac{p^3}{3!}\big\{\mathbf{v}^3(\boldsymbol{x}) + O(p)\big\} \\
= \boldsymbol{x} &+ p\mathbf{u}(\boldsymbol{x}) + \frac{p^2}{2}\mathbf{u}^2(\boldsymbol{x}) + \frac{p^3}{3!}\mathbf{u}^3(\boldsymbol{x}) + O(p^4) \\
&+ p\Big\{\mathbf{v}(\boldsymbol{x}) + p\mathbf{u}\big(\mathbf{v}(\boldsymbol{x})\big) + \frac{p^2}{2}\mathbf{u}^2\big(\mathbf{v}(\boldsymbol{x})\big) + O(p^3)\Big\} \\
&+ \frac{p^2}{2}\big\{\mathbf{v}^2(\boldsymbol{x}) + p\mathbf{u}\big(\mathbf{v}^2(\boldsymbol{x})\big) + O(p^2)\big\} + \frac{p^3}{3!}\big\{\mathbf{v}^3(\boldsymbol{x}) + O(p)\big\} ,
\end{aligned}$$

so that

$$\begin{aligned}
\boldsymbol{z} = \boldsymbol{x} &+ p\big\{\mathbf{v}(\boldsymbol{x}) + \mathbf{u}(\boldsymbol{x})\big\} + \frac{p^2}{2}\big\{\mathbf{v}^2(\boldsymbol{x}) + 2\mathbf{u}\big(\mathbf{v}(\boldsymbol{x})\big) + \mathbf{u}^2(\boldsymbol{x})\big\} + \\
&+ \frac{p^3}{3!}\big\{\mathbf{v}^3(\boldsymbol{x}) + \mathbf{u}^3(\boldsymbol{x}) + 3\mathbf{u}^2\big(\mathbf{v}(\boldsymbol{x})\big) + 3\mathbf{u}\big(\mathbf{v}^2(\boldsymbol{x})\big)\big\} + O(p^4) .
\end{aligned} \tag{46}$$

Now we rewrite (43) in the form

$$\boldsymbol{z} = \eta^1(\boldsymbol{x}) = \boldsymbol{x} + \mathbf{w}(\boldsymbol{x}) + \frac{1}{2}\mathbf{w}^2(\boldsymbol{x}) + \frac{1}{3!}\mathbf{w}^3(\boldsymbol{x}) + \cdots , \tag{47}$$

and we use (44) for $\mathbf{w}(\boldsymbol{x})$. Then (when omitting $\boldsymbol{x}$)

$$\begin{aligned}
\mathbf{w}^2 &= (\mathbf{w}_1 + \mathbf{w}_2 + \mathbf{w}_3 + \cdots)(\mathbf{w}_1 + \mathbf{w}_2 + \mathbf{w}_3 + \cdots) \\
&= \mathbf{w}_1^2 + \mathbf{w}_1(\mathbf{w}_2) + \mathbf{w}_1(\mathbf{w}_3) + \cdots + \mathbf{w}_2(\mathbf{w}_1) + \mathbf{w}_2^2 + \cdots + \mathbf{w}_3(\mathbf{w}_1) + \cdots
\end{aligned} \tag{48}$$

$$\begin{aligned}
\mathbf{w}^3 &= \mathbf{w}(\mathbf{w}^2) = (\mathbf{w}_1 + \mathbf{w}_2 + \cdots)\big(\mathbf{w}_1^2 + \mathbf{w}_1(\mathbf{w}_2) + \mathbf{w}_2(\mathbf{w}_1) + \cdots\big) \\
&= \mathbf{w}_1^3 + \mathbf{w}_1^2(\mathbf{w}_2) + \mathbf{w}_2(\mathbf{w}_1^2) + \cdots .
\end{aligned} \tag{49}$$

From (48) and (49) we put into (47) and we get

$$\begin{aligned} \boldsymbol{z} = \boldsymbol{x} + (\mathbf{w}_1 + \mathbf{w}_2 + \mathbf{w}_3 + \cdots) + \frac{1}{2}\big[\mathbf{w}_1^2 + \mathbf{w}_1(\mathbf{w}_2) + \mathbf{w}_2(\mathbf{w}_1) + \cdots\big] \\ + \frac{1}{3!}[\mathbf{w}_1^3 + \cdots] + O(p^4)\,. \end{aligned} \tag{50}$$

By comparing (46) and (50) and using (45) we get

$$p^1: \quad \mathbf{w}_1 = \mathbf{v} + \mathbf{u}\,, \tag{51}$$

$$p^2: \quad \mathbf{w}_2 + \frac{1}{2}\mathbf{w}_1^2 = \frac{1}{2}\big\{\mathbf{v}^2 + 2\mathbf{u}(\mathbf{v}) + \mathbf{u}^2\big\}\,, \tag{52}$$

$$p^3: \quad \mathbf{w}_3 + \frac{1}{2}\mathbf{w}_1(\mathbf{w}_2) + \frac{1}{2}\mathbf{w}_2(\mathbf{w}_1) + \frac{1}{3!}\mathbf{w}_1^3 = \frac{1}{3!}\big(\mathbf{v}^3 + 3\mathbf{u}^2(\mathbf{v}) + 3\mathbf{u}(\mathbf{v}^2) + \mathbf{u}^3\big)\,. \tag{53}$$

Now we put from (51) into (52) using

$$\mathbf{w}_1^2 = (\mathbf{v}+\mathbf{u})(\mathbf{v}+\mathbf{u}) = \mathbf{v}^2 + \mathbf{v}(\mathbf{u}) + \mathbf{u}(\mathbf{v}) + \mathbf{u}^2 \tag{54}$$

and we get

$$\mathbf{w}_2 = \frac{1}{2}\big(\mathbf{u}(\mathbf{v}) - \mathbf{v}(\mathbf{u})\big) = -\frac{1}{2}[\mathbf{v},\mathbf{u}] = \frac{1}{2}[\mathbf{v},\mathbf{u}]^{\sim}\,, \tag{55}$$

where we use the modified Lie bracket $[\cdot,\cdot]^{\sim}$ from Remark 5.1. Now we put into (53) using

$$\begin{aligned} \mathbf{w}_1^3 &= \mathbf{w}_1(\mathbf{w}_1^2) = (\mathbf{v}+\mathbf{u})\big(\mathbf{v}^2 + \mathbf{v}(\mathbf{u}) + \mathbf{u}(\mathbf{v}) + \mathbf{u}^2\big) \\ &= \mathbf{v}^3 + \mathbf{v}^2(\mathbf{u}) + \mathbf{v}\big(\mathbf{u}(\mathbf{v})\big) + \mathbf{v}(\mathbf{u}^2) + \mathbf{u}(\mathbf{v}^2) + \mathbf{u}\big(\mathbf{v}(\mathbf{u})\big) + \mathbf{u}^2(\mathbf{v}) + \mathbf{u}^3\,; \end{aligned} \tag{56}$$

$$\mathbf{w}_1(\mathbf{w}_2) = (\mathbf{v}+\mathbf{u})\Big(\frac{1}{2}\big(\mathbf{u}(\mathbf{v}) - \mathbf{v}(\mathbf{u})\big)\Big) = \frac{1}{2}\big(\mathbf{v}^2(\mathbf{u}) - \mathbf{v}\big(\mathbf{u}(\mathbf{v})\big) + \mathbf{u}\big(\mathbf{v}(\mathbf{u})\big) - \mathbf{u}^2(\mathbf{v})\big); \tag{57}$$

$$\mathbf{w}_2(\mathbf{w}_1) = \frac{1}{2}\big(\mathbf{v}(\mathbf{u}) - \mathbf{u}(\mathbf{v})\big)(\mathbf{v}+\mathbf{u}) = \frac{1}{2}\big(\mathbf{v}\big(\mathbf{u}(\mathbf{v})\big) + \mathbf{v}(\mathbf{u}^2) - \mathbf{u}(\mathbf{v}^2) - \mathbf{u}\big(\mathbf{v}(\mathbf{u})\big)\big)\,. \tag{58}$$

After putting (56), (57) and (58) into (53) and simplifying we get

$$\mathbf{w}_3 = \frac{1}{12}\Big(\big[\mathbf{v},[\mathbf{v},\mathbf{u}]\big] - \big[\mathbf{u},[\mathbf{v},\mathbf{u}]\big]\Big) = \frac{1}{12}\Big(\big[\mathbf{v},[\mathbf{v},\mathbf{u}]^{\sim}\big]^{\sim} - \big[\mathbf{u},[\mathbf{v},\mathbf{u}]^{\sim}\big]^{\sim}\Big)\,. \tag{59}$$

This result suggests the following hypothesis:

Hypothesis. One of the vector fields satisfying (38), i.e. $\mathrm{Exp}(\mathbf{w}) = \mathrm{Exp}(\mathbf{v}) \circ \mathrm{Exp}(\mathbf{u})$, is the vector field $\mathbf{w}(\mathbf{v}:\mathbf{u})$ given by the following BCH formula

$$\mathbf{w}(\mathbf{v}:\mathbf{u}) = \mathbf{v} + \mathbf{u} + \frac{1}{2}[\mathbf{v},\mathbf{u}]^{\sim} + \frac{1}{12}\Big(\big[\mathbf{v},[\mathbf{v},\mathbf{u}]^{\sim}\big]^{\sim} - \big[\mathbf{u},[\mathbf{v},\mathbf{u}]^{\sim}\big]^{\sim}\Big) + \cdots$$

and the convergence of the right-hand side implies that the diffeomorphism $\psi^1 \circ \varphi^1$ is embeddable.

Remark 7.1. It is useful to note that our hypothesis is also supported by the result from the Paragraph 5, because the result (27) from Remark 5.1, obtained by a fully rigorous way, agrees with our hypothesis for linear vector fields.

8. Numerical experiment

As mentioned in the Remark 7.1 the full proof of our Hypothesis is not complete in the sense that the problems with the convergence in the BCH formula are not yet solved.

Because of this, we have made the following numerical experiment to shed more light on the relation between the ω-limit behavior of the orbits of period mapping P and the ω-limit behavior of the trajectories of the vector field $\mathbf{w}(\mathbf{v}:\mathbf{u})$.

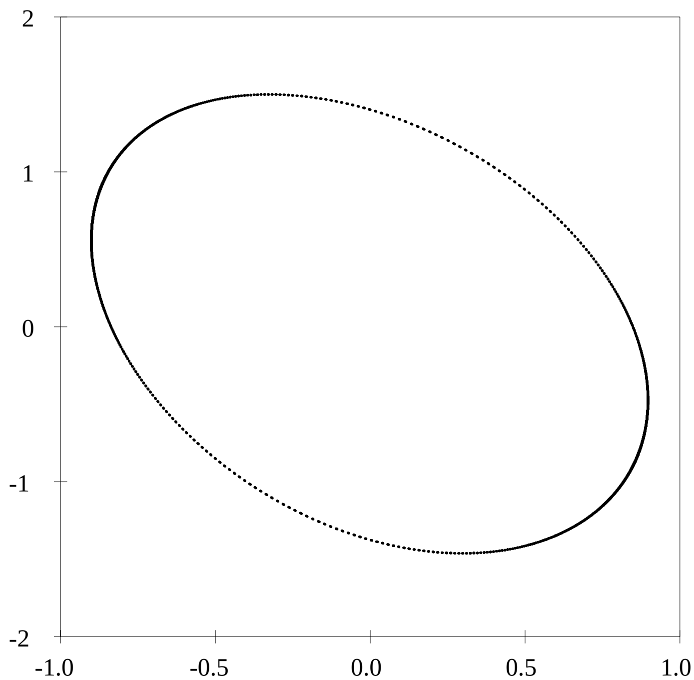

FIGURE 3. Invariant curve of the period map P for the zig-zag dynamical system "blinking cycles".

Let us have two planar vector fields

$$\mathbf{u}(x,y) = \begin{bmatrix} \mu(x-1) - y - (x-1)\big[(x-1)^2 + y^2\big] \\ (x-1) + \mu y - y\big[(x-1)^2 + y^2\big] \end{bmatrix}, \tag{60}$$

$$\mathbf{v}(x,y) = \begin{bmatrix} \mu(x+1) - y - (x+1)\big[(x+1)^2 + y^2\big] \\ (x+1) + \mu y - y\big[(x+1)^2 + y^2\big] \end{bmatrix} \tag{61}$$

each of which has a stable limit cycle with the radius $\sqrt{\mu}$ and the center $(1;0)$, $(-1;0)$ respectively. Consider now the zig-zag dynamical system (2) with these vector fields for the switching period $p = 0.02$, where $\mu = 3.5$.

Then the corresponding period map P has the invariant closed curve depicted in Fig. 3.

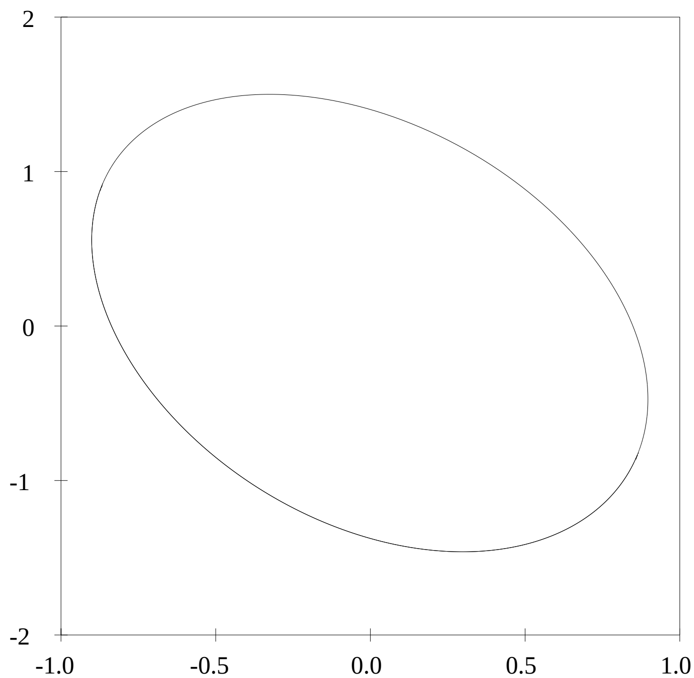

FIGURE 4. To approximate the dynamics of the zig-zag system "blinking cycles" shown in Fig. 3, we use the first 1 to 4 terms of the BCH formula. When taking the first 1 or 2 terms the resulting dynamical system has a qualitatively different attractor (a stable stationary point) not shown here. When taking the first 3 or 4 terms the resulting dynamical system has a stable limit cycle that is almost indistinguishable from the invariant curve of the original zig-zag system. Cf. Fig. 3.

Now we construct using (60) and (61) the vector fields

$$\begin{aligned}
\mathbf{w}_1 &= \mathbf{v} + \mathbf{u}\,, \\
\mathbf{w}_2 &= \frac{1}{2}[\mathbf{v}, \mathbf{u}]^{\sim}\,, \\
\mathbf{w}_3 &= \frac{1}{12}\left[\mathbf{v} - \mathbf{u}, [\mathbf{v}, \mathbf{u}]^{\sim}\right]^{\sim}\,, \\
\mathbf{w}_4 &= -\frac{1}{24}\left[\mathbf{v}, \left[\mathbf{u}, [\mathbf{v}, \mathbf{u}]^{\sim}\right]^{\sim}\right]^{\sim}\,,
\end{aligned}$$

i.e. these vector fields are successive terms in the BCH formula. The computation of $\mathbf{w}_k$ was made using the computer algebra system *Mathematica*. Further we construct the vector fields

$$\mathbf{w}_{1,1} = \mathbf{w}_1 ,$$
$$\mathbf{w}_{1,2} = \mathbf{w}_1 + \mathbf{w}_2 ,$$
$$\mathbf{w}_{1,3} = \mathbf{w}_1 + \mathbf{w}_2 + \mathbf{w}_3 ,$$
$$\mathbf{w}_{1,4} = \mathbf{w}_1 + \mathbf{w}_2 + \mathbf{w}_3 + \mathbf{w}_4 ,$$

and we solve numerically the differential equations

$$\dot{\boldsymbol{x}} = \mathbf{w}_{1,i}(\boldsymbol{x}) , \qquad i = 1, 2, 3, 4 . \tag{62}$$

The first two systems have only stable steady states, while the last two systems have an almost indistinguishable stable limit cycle depicted in Fig. 4. The reader can convince himself that the coincidence of the invariant curve from Fig. 3 with the closed trajectory in Fig. 4 is surprisingly good.

This demonstrates clearly how the BCH formula can be used to approximate the dynamics of a zig-zag dynamical system by an autonomous system. The BCH formula gives an essential refinement of the averaging method studied in [5].

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