ISSN 1560-3547, Regular and Chaotic Dynamics, 2008, Vol. 13, No. 3, pp. 155–165. © Pleiades Publishing, Ltd., 2008.

Stability Condition for Vertical Oscillation of 3-dim Heavy Spring Elastic Pendulum

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Received February 26, 2008; accepted April 28, 2008

Abstract—Equations of motion for 3-dim heavy spring elastic pendulum are derived and rescaled to contain a single parameter. Condition for the stability of vertical large amplitude oscillations is derived analytically relating the parameter of the system and the amplitude of the vertical oscillation. Numerical continuation is used to find the border of the stability region in parameter space with high precision. The stability condition is approximated by a simple formula valid for a large range of the parameter and of the amplitude of oscillation. The bifurcation responsible for the loss of stability is identified.

MSC2000 numbers: 70G60, 70K20, 37J25, 37J45

DOI: 10.1134/S1560354708030027

Key words: elastic pendulum, stability condition

1. INTRODUCTION

The first known publication about the elastic pendulum is the paper [1] by Vitt and Gorelik. They consider small amplitude oscillation of a planar elastic pendulum in the 2:1 resonance. Connection to the Fermi resonance of CO_2 is mentioned. This paper has been translated from Russian by Lisa Shields and published by Peter Lynch on his web page.

Olsson in [2] excludes more than one parametric resonance. Cayton in [3] uses numerical simulation and observes that in 3D "the swinging plane rotates by an (apparently arbitrary) angle". Relation to 3-wave interaction in nonlinear media is mentioned.

Rusbridge in [4] uses a light bulb with a battery attached to the bob of the elastic pendulum and a film to record the motion of the bob and comes to the conclusion that "some pendulums seem very bad for no obvious reasons (rotation of the swinging plane)". Breitenberger and Mueller in [5] derive two different criteria for the stability of the suspension mode.

Lai in [6] compares his analytical results for the slow varying amplitude and phases with his experimental results. This is the only paper that briefly discusses the mass of the spring concluding that the finite mass of the spring should not affect the results.

Deterministic chaos has been observed in numerical simulation and presented in the form of Poincaré section, auto-correlation function, Lyapunov exponent and power spectrum in [7–9].

In [10] Anicin *et al.* study the stability of small amplitude vertical oscillation by means of a locus line drawn in the Ince-Strut stability chart of the respective Matthieu equation. They determine graphically the range of mass leading to instability for a particular amplitude of the initially vertical oscillation and the corresponding growth coefficient.

Davidovic *et al.* in [11] use parabolic coordinates to find the parabolas bounding the accessible domain in the x-y plane for small amplitude oscillation in the 2:1 resonance. A part of the bifurcation diagram of a plane elastic pendulum is sketched in [12].

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Monodromy of an integrable approximation of the elastic pendulum is studied in [13]. The most thorough treatment of small amplitude oscillation of both plane and space elastic pendulum can be found in the works by Peter Lynch [14–19].

This paper is organized as follows. In Section 2 we define the system, derive equations of motion and rescale them to get equations with a single parameter. In Section 3 we study the stability of the linearized system using variational equations. The condition for the border of stable and unstable regions in the parameter space is formulated. Numerical continuation is used to find this border with a high precision. The resulting data are transformed and fitted to give a simple closed form formula for stability. The accuracy of this formula is given. In conclusion we compare our method to the classical approach based on Taylor approximation.

2. DEFINITION OF THE SYSTEM

Consider a pendulum consisting of a point bob of mass m_B suspended on a homogeneous elastic spring of mass m_S with the elasticity constant k, see Fig. 1. Assume a homogeneous gravitational field with intensity $(0, 0, -g)$. Denoting the coordinates of the bob X, Y and Z, and assuming the spring is stretched homogeneously, the kinetic energy of the system bob+spring is

$$E_{kin} = \frac{1}{2}\Big(m_B + \frac{m_S}{3}\Big)V^2$$

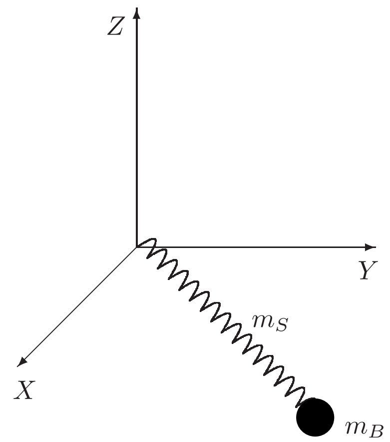

Fig. 1. Heavy spring elastic pendulum consists of a point mass m_B attached to a spring with mass m_S which is fixed at the other end point.

where V is the velocity of the bob. The potential energy is

$$E_{pot} = \Big(m_B + \frac{m_S}{2}\Big)gZ + \frac{k}{2}(\sqrt{X^2 + Y^2 + Z^2} - \ell_0)^2$$

where ℓ_0 is the length of the unloaded spring. Then the equations of motion are

$$\begin{aligned}
(m_B + \frac{m_S}{3})\frac{d^2X}{dT^2} &= k\Big(\frac{\ell_0}{\sqrt{X^2 + Y^2 + Z^2}} - 1\Big)X, \\
(m_B + \frac{m_S}{3})\frac{d^2Y}{dT^2} &= k\Big(\frac{\ell_0}{\sqrt{X^2 + Y^2 + Z^2}} - 1\Big)Y, \\
(m_B + \frac{m_S}{3})\frac{d^2Z}{dT^2} &= k\Big(\frac{\ell_0}{\sqrt{X^2 + Y^2 + Z^2}} - 1\Big)Z - (m_B + \frac{m_S}{2})g.
\end{aligned}$$

It is convenient to introduce dimensionless variables x, y, z, t by $x = \frac{X}{\ell_0}$, $y = \frac{Y}{\ell_0}$, $z = \frac{Z}{\ell_0}$, $t = T\sqrt{\frac{k}{m_B + \frac{m_S}{3}}}$ because then the equations of motion are

$$\begin{aligned}
\ddot{x} &= \Big(\frac{1}{\sqrt{x^2+y^2+z^2}} - 1\Big)x,\\
\ddot{y} &= \Big(\frac{1}{\sqrt{x^2+y^2+z^2}} - 1\Big)y,\\
\ddot{z} &= \Big(\frac{1}{\sqrt{x^2+y^2+z^2}} - 1\Big)z - p,
\end{aligned} \tag{1}$$

where upper dot means the derivative with respect to time t. This system has the only parameter

$$p = \frac{(m_B + \frac{m_S}{2})g}{k\ell_0}.$$

The parameter p has several interpretations: it is the (dimensionless) strength of the external field; it is the relative prolongation

$$p = \frac{\ell - \ell_0}{\ell_0}$$

of the spring due to the effective load $m_B + \frac{m_S}{2}$ and it is related to

$$q = \frac{p}{1+p} = \frac{\ell - \ell_0}{\ell} = \left(\frac{\Omega_P}{\Omega_S}\right)^2, \tag{2}$$

where

$$\Omega_P = \sqrt{\frac{g(m_B + \frac{m_S}{2})}{\ell(m_B + \frac{m_S}{3})}}$$

is the frequency of the pendulum (or quasi-horizontal) motion and

$$\Omega_S = \sqrt{\frac{k}{m_B + \frac{m_S}{3}}}$$

is the frequency of the spring (or vertical) motion. Note that $p > 0$ is equivalent to $0 < q < 1$ i.e. $\Omega_P < \Omega_S$. Also note that the mass-less spring elastic pendulum (often discussed in the literature) is a special case with $m_S = 0$ included in our more general case.

The system (1) is conservative, the total dimensionless energy

$$e = \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \frac{1}{2}(\sqrt{x^2+y^2+z^2} - 1)^2 + pz$$

is conserved. An example of the potential energy for $p = 0.3$ is shown in Fig. 2.

3. STABILITY OF VERTICAL OSCILLATION

By introducing six variables

$$(x_1, x_2, x_3, x_4, x_5, x_6) = (x, \dot{x}, y, \dot{y}, z, \dot{z})$$

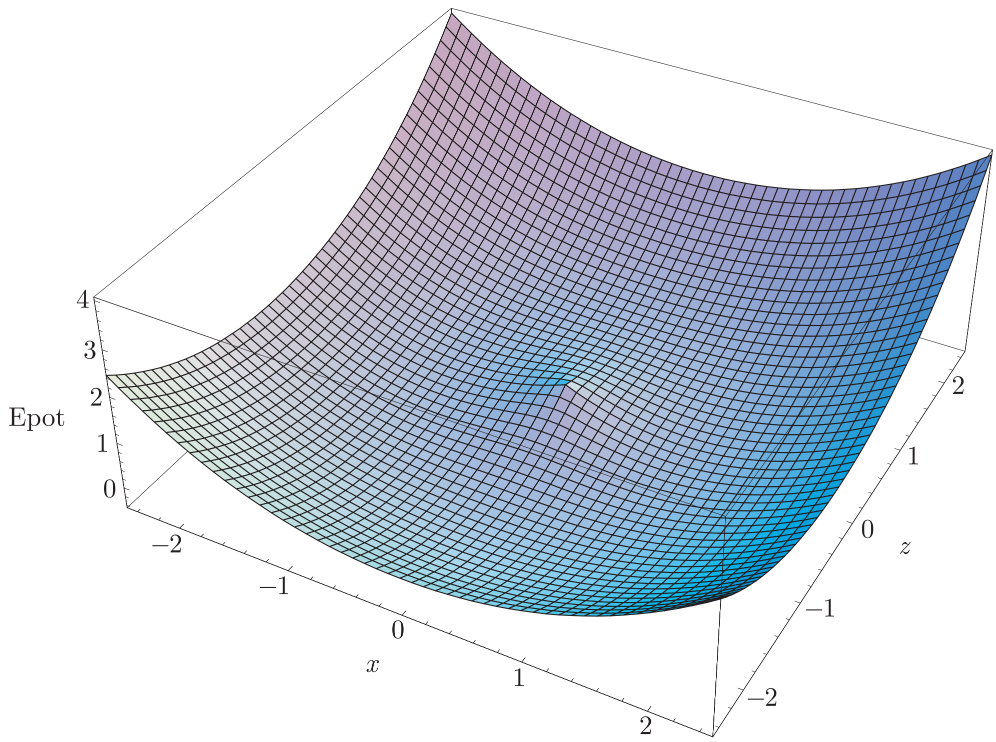

Fig. 2. The potential well of the elastic pendulum for $p = 0.3$ as a function of x and z for $y = 0$.

we rewrite the system (1) into a system of six equations of the first order

$$\begin{aligned} \dot{x_1} &= x_2, \\ \dot{x_2} &= \Big(\frac{1}{\sqrt{x_1^2 + x_3^2 + x_5^2}} - 1\Big)x_1, \\ \dot{x_3} &= x_4, \\ \dot{x_4} &= \Big(\frac{1}{\sqrt{x_1^2 + x_3^2 + x_5^2}} - 1\Big)x_3, \\ \dot{x_5} &= x_6, \\ \dot{x_6} &= \Big(\frac{1}{\sqrt{x_1^2 + x_3^2 + x_5^2}} - 1\Big)x_5 - p. \end{aligned} \tag{3}$$

There are exactly two equilibrium points of this system: one stable (a center) $(0, 0, 0, 0, -1 - p, 0)$ with the energy $e = -p - \frac{p^2}{2}$ and the other unstable (a saddle) $(0, 0, 0, 0, 1 - p, 0)$ with the energy $e = p - \frac{p^2}{2}$ (this one having physical meaning only for $p < 1$). This suggests another interpretation of the parameter p: it is half of the difference between the energies of the two equilibrium points of the system.

We want to investigate the stability of the vertical oscillation

$$(x_1, x_2, x_3, x_4, x_5, x_6) = (0, 0, 0, 0, -1 - p + a \sin t, a \cos t) \tag{4}$$

of system (3), where a is the amplitude of the vertical oscillation.

Denoting f the right hand side of (3), the stability of a solution $x(t)$ of $\dot{x} = f(x)$ can be found by investigating eigenvalues of the matrix $M(t)$ where

$$\dot{M}(t) = f'(x(t)) \cdot M(t) \tag{5}$$

(the variational equation) with the initial condition $M(0) = E$ where E is the identity matrix and $f'_{ij} = \frac{\partial f_i}{\partial x_j}$ is the matrix of partial derivatives of f.

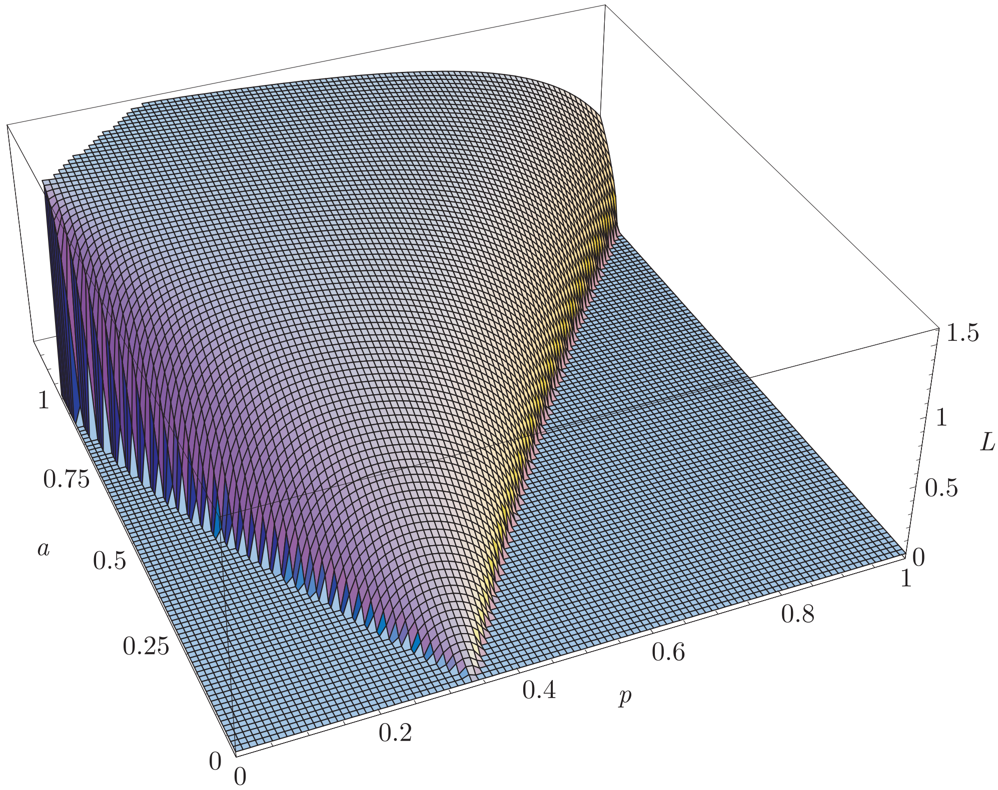

Fig. 3. Rescaled maximum of absolute values of eigenvalues of the matrix $M(2\pi)$ as a function of the strength of the external field p and the amplitude of the periodic solution a. The two plateaus indicate a stable region and the hill between them shows an unstable region. The value plotted is $L = \arctan(\max_i |\lambda_i| - 1)$ to keep the values finite.

In our case the matrix f' evaluated in the periodic solution (4) is

$$f' = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ f'_{21} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & f'_{43} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix} \tag{6}$$

where

$$f'_{21} = f'_{43} = \frac{1}{|-1-p+a\sin t|} - 1.$$

This brings a condition to the amplitude a

$$|a| < 1 + p. \tag{7}$$

Note that this is much more general than the condition $|a| \ll 1$ which is often used in the literature.

In a more general case when the potential $E_{pot} = \frac{1}{2}(\sqrt{x^2+y^2+z^2}-1)^2 + pz$ is replaced by $E_{pot} = s(r) + pz$ with $r = \sqrt{x^2+y^2+z^2}$ where $s(r)$ is the spherical component of the potential, we get

$$f'_{21} = f'_{43} = -\frac{s'(r)}{r}.$$

Now we can integrate (3) and (5) for one period $\tau = 2\pi$ to find the six eigenvalues $\lambda_i(p,a)$, $i = 1 \dots 6$ of the matrix $M(2\pi)$. These are six complex functions of two real arguments: the strength of the external field p and the amplitude of the periodic oscillation a.

Actually we do not have to integrate (3) because we know the solution (4) and we do not have to integrate 36 equations (5) because (6) splits into three 2×2 blocks. It is sufficient to integrate 4 linear non-autonomous equations corresponding to the upper left block of (6).

Figure 3 shows the dependence of the rescaled maximum of absolute values of eigenvalues $L = \arctan(\max_i |\lambda_i| - 1)$ as a function of p (the strength of the external field) and a (the amplitude of the vertical periodic solution). As $\max_i |\lambda_i| \to \infty$ for $a \to 1+p$, the function $\arctan(x-1)$ is used to keep the values finite. In this figure, we can identify two plateaus with $\max_i |\lambda_i| = 1$ and a hill between them with $\max_i |\lambda_i| > 1$. The hill corresponds to unstable solution. A small initial perturbation will grow approximately exponentially (at least for some time) as there is one eigenvalue (let us call it λ_1) outside the unit circle in the complex plane. What are the remaining five eigenvalues? As the eigenvalues come in pairs with reciprocal absolute values (for Hamiltonian systems) there is another eigenvalue λ_2 with absolute value less than 1. The next pair of eigenvalues equals to the first one $\lambda_3 = \lambda_1$, $\lambda_4 = \lambda_2$. The remaining two eigenvalues are equal to 1 exactly, one corresponding to the direction of the trajectory and the last one corresponding to the direction perpendicular to the surface of constant total energy.

The two plateaus correspond to periodic solution that is orbitally stable with $|\lambda_i| = 1$ for $i = 1 \dots 6$.

Our main interest is the border between the two plateaus corresponding to stable oscillation and the hill corresponding to unstable solution. This border is a V-shaped curve in the p-a plane. We can describe this curve as $a = |a_C(p)|$ where $a_C(p)$ is a function to be found. For a fixed p it gives a critical value $|a_C|$, such that for $|a| < |a_C|$ the periodic solution is stable and for $|a| > |a_C|$ it is unstable. We want to find this function $a_C(p)$. As λ_i is an even function of a we can choose the sign of $a_C(p)$ to be negative for small p and positive for large p, so that $a_C(p)$ is a smooth function.

When crossing this border from stable to unstable region then first one pair of eigenvalues of $M(2\pi)$ lies on the unit circle before the crossing, i.e. $\lambda_{1,2} = \exp(\pm i\phi)$, then they are equal to $\lambda_{1,2} = -1$ for the critical value, and finally they turn to two distinct real values $\lambda_1 > -1$, $\lambda_2 < -1$. This is accompanied by a period-doubling bifurcation. A small homoclinic orbit (figure eight) to the origin appears in the Poincaré section (this corresponds to a pinched torus for the continuous time system).

This is a kind of parametric resonance, where the vertical oscillation is considered as the periodic change of the parameter (the length of the pendulum) which results in unstability in other directions.

We can describe this border by the condition

$$D(p,a) = 0$$

where

$$D = (M_{11} + M_{22})^2 - 4$$

is the discriminant of the characteristic equation of the upper left block of $M(2\pi)$ (using also the fact, that the determinant of this block is equal to 1). The discriminant as a function of p and a is shown in Fig. 4. Actually the value $\arctan(D)$ is plotted to keep the values finite. We will detect the border of stability by the condition $D(p,a) = 0$, where $D(p,a)$ is a smooth function. This is much more convenient than to consider the sharp edge between the plateau and the hill in Fig. 3.

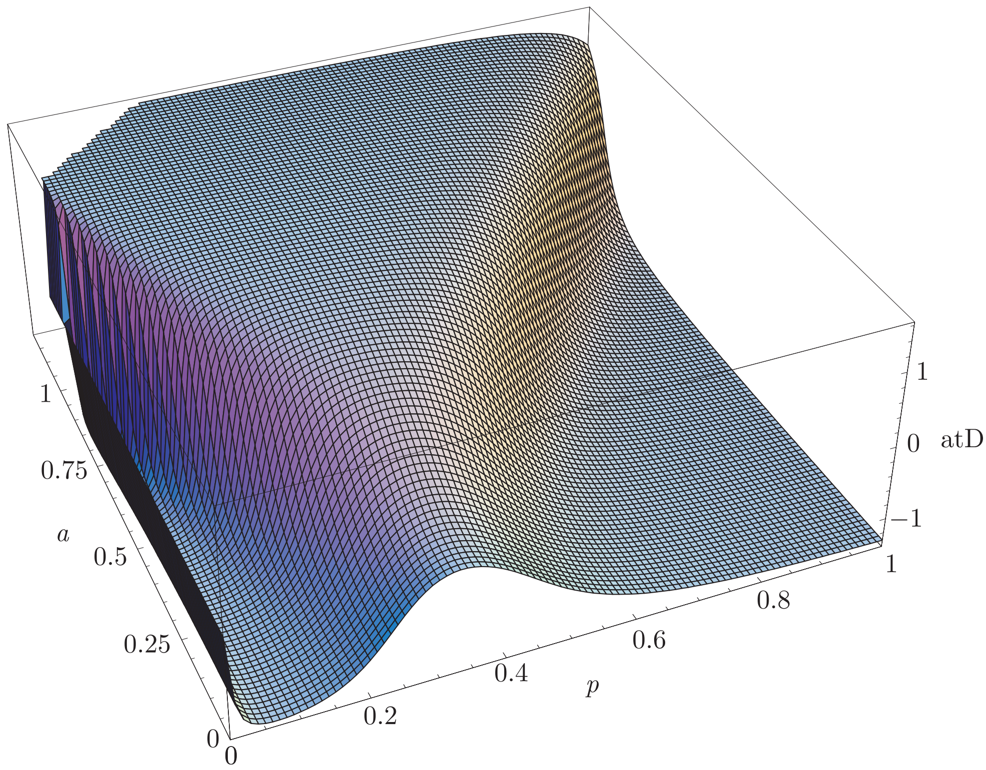

Fig. 4. Rescaled discriminant of the characteristic equation for the matrix $M(2\pi)$ as a function of the strength of the external field p and the amplitude of the periodic solution a. This is a suitable test function to detect the border between stable and unstable regions in the p-a plane. The value plotted is $atD = \arctan(D)$ to keep the values finite.

The graph of $a_C(p)$ crosses the p axis in a point somewhere between $p = 0.2$ and $p = 0.4$, see Fig. 3. We can find this value exactly. If $a = 0$ then

$$f'_{21} = \frac{1}{1+p} - 1 = -\frac{p}{1+p} = -q$$

is independent of time t and the variational equation

$$\dot{M} = f' \cdot M$$

with the upper left 2x2 block

$$f' = \begin{pmatrix} 0 & 1 \\ -q & 0 \end{pmatrix}$$

has the solution

$$\begin{aligned} M(2\pi) &= \exp\left(2\pi \begin{pmatrix} 0 & 1 \\ -q & 0 \end{pmatrix}\right) \\ &= \begin{pmatrix} \cos(2\pi\sqrt{q}) & \frac{1}{\sqrt{q}}\sin(2\pi\sqrt{q}) \\ -\sqrt{q}\sin(2\pi\sqrt{q}) & \cos(2\pi\sqrt{q}) \end{pmatrix}. \end{aligned}$$

Then the discriminant of the characteristic equation is

$$D = (M_{11} + M_{22})^2 - 4 = (2\cos(2\pi\sqrt{q}))^2 - 4 = -4\sin^2\left(2\pi\sqrt{\frac{p}{p+1}}\right)$$

and the equation of the border

$$D(p, 0) = 0$$

has for $p > 0$ the only solution

$$p_{res} = \frac{1}{3} \tag{8}$$

corresponding to

$$q_{res} = \frac{1}{4}.$$

In other words, recalling (2), when the ratio of the two frequencies is

$$\frac{\Omega_P}{\Omega_S} = \frac{1}{2},$$

then arbitrarily small vertical oscillation is unstable, which is called 1:2 resonance.

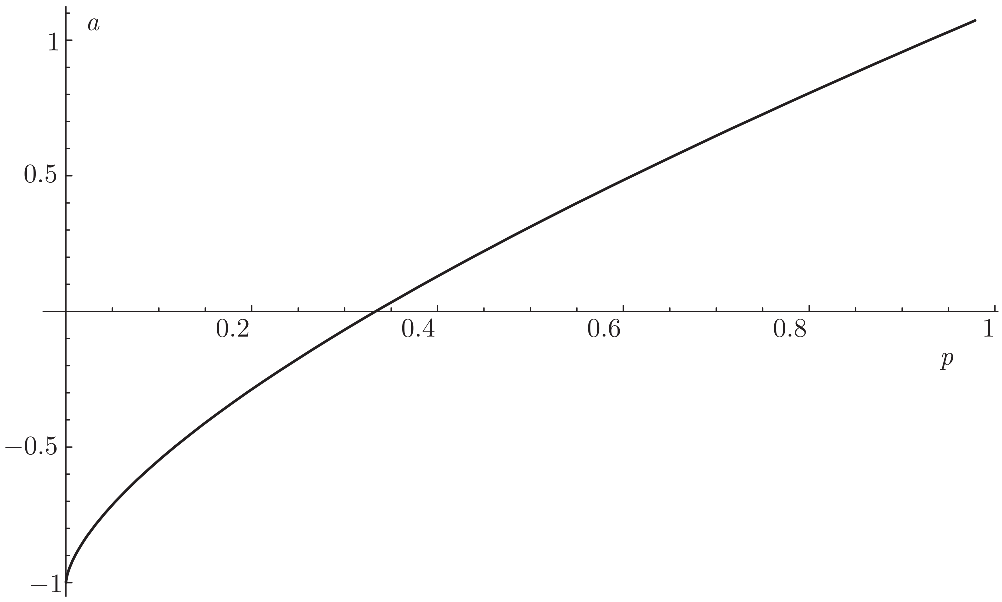

Fig. 5. The function $a_C(p)$ giving the critical value such that the vertical oscillation with the amplitude a is stable if $|a| < |a_C(p)|$.

This is in agreement with our expectation based on Fig. 3. Also, this excludes other resonances which one might expect for frequencies Ω_P and Ω_S in a ratio of small integers.

To prove this, for each point (p, a) in the p-a plane with $p > 0$, $p \neq \frac{1}{3}$, $a = 0$ we have $D(p, a) < 0$. Then we can find a neighborhood of this point such that $D(p, a) < 0$ in this neighborhood as $D(p, a)$ is continuous in this point. Negative discriminant (together with determinant equal to 1) implies two imaginary eigenvalues on the unit circle and thus no eigenvalue with the absolute value greater than 1.

To find the function giving the critical value $a_C(p)$ we can use continuation, an efficient method based on the predictor–corrector algorithm. It has the advantage, due to the corrector step, that the error does not accumulate like in numerical solution of ODE. The result is presented in Fig. 5. The curve crosses the p axis in $p = \frac{1}{3}$ in agreement with (8). The shape of the curve $a_C(p)$ resembles that of the square root, shifted one unit down. To find the correct exponent we plot $\log(a + 1)$ vs $\log(p)$ in Fig. 6 together with a linear fit. As you can see the agreement is surprising, all the points (computed with the error of order 10^{-16}) lie on a straight line (with a deviation not observable

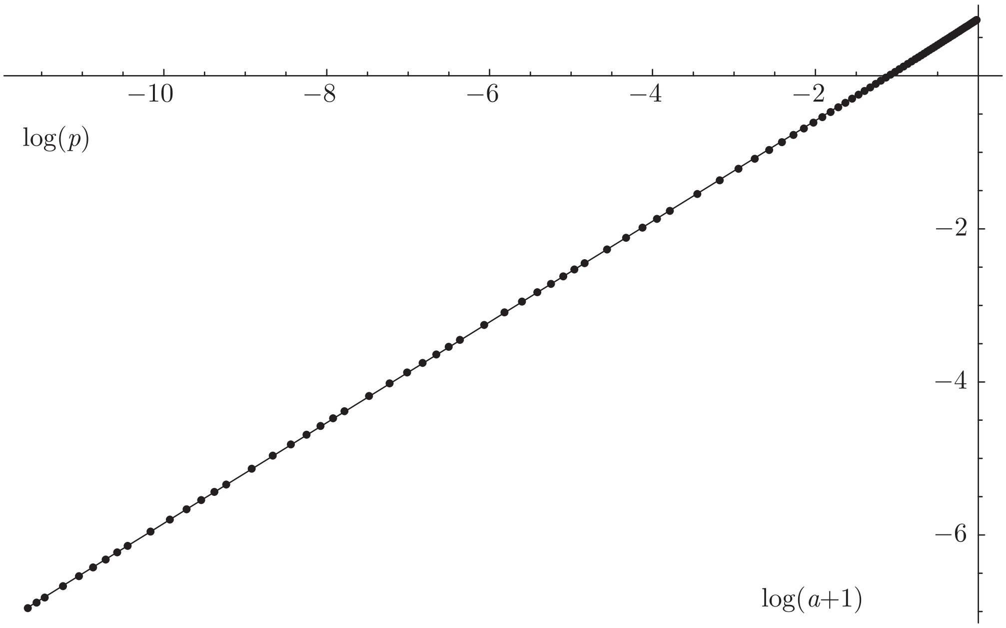

Fig. 6. The dependence of the critical value of the amplitude of vertical oscillation a on the parameter p (as in Fig. 5) in a log–log plot (after a shift by a). The circles represent the data computed with a high accuracy (typically 10^{-16}). The line is a least squares fit.

in the graph). The linear fit gives the slope in the log-log plot (and thus the exponent of p) to be approximately $\frac{2}{3}$. Thus we can approximate the critical value $a_C(p)$ by a fitted one $a_F(p)$ in the form

$$a_C(p) \doteq a_F(p) = (cp)^{\frac{2}{3}} - 1$$

where the constant c can be found from the condition that the curve must cross the p axis in the point $p = \frac{1}{3}$ exactly. This gives $c = 3$ and the conclusion can be formulated in this

Theorem. *The vertical oscillation of the* 3*-dim heavy spring elastic pendulum with the relative amplitude a is stable for*

$$|a| < |a_C(p)|$$

where

$$a_C(p) \doteq (3p)^{\frac{2}{3}} - 1 \tag{9}$$

with the parameter

$$p = \frac{(m_B + \frac{m_S}{2})g}{k\ell_0}$$

where m_B is the mass of the bob, m_S is the mass of the spring, k is the spring constant, g is the acceleration due to gravity and ℓ_0 is the length of the free spring. The unit of the amplitude a is the length ℓ_0 of the free spring. The error of this estimate of $a_C(p)$ is less than 0.005 *for* $0 < p < 0.6$.

Unfortunately, this approximation is valid only for small p. For large p a similar approach gives the estimate

$$a_C(p) \doteq 1 + p - \frac{5}{p^2}.$$

The error of this estimate of $a_C(p)$ is less than 0.03 for $p > 4$.

Figure 7 shows the regions in the p-a plane with stable (light shade) and unstable (dark shade) vertical oscillation with the amplitude a and the parameter p.

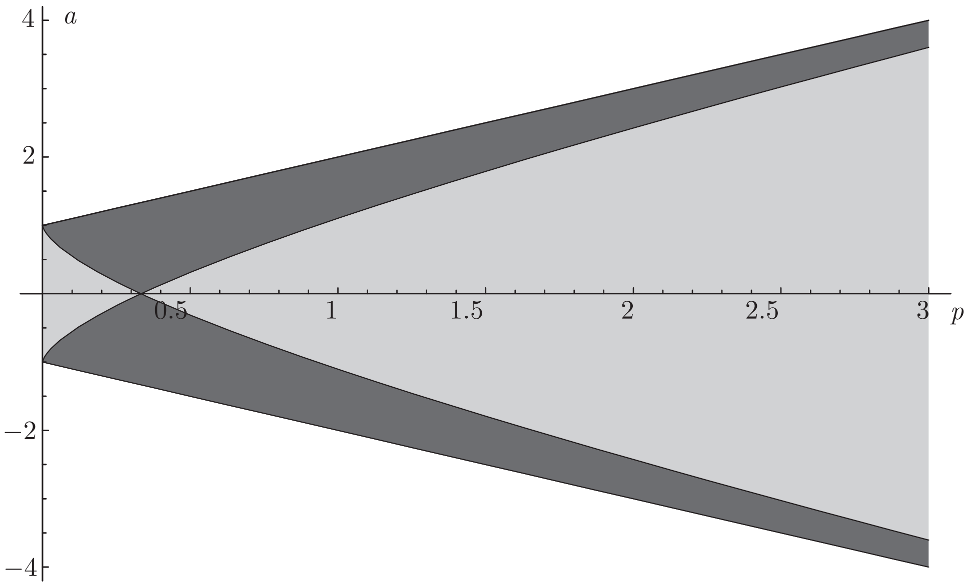

Fig. 7. The stable region (light shade) and the unstable region (dark shade) for the vertical oscillation with the amplitude a and the external field p for $p > 0$ and $|a| < 1 + p$ (only points with $p < 3$ are shown in the graph for clarity reason).

4. CONCLUSION

The elastic pendulum is often studied under the assumption of a mass-less spring. We derived the equations of motion for the elastic pendulum with a spring of arbitrary mass and we rescaled the equations to contain a single parameter.

Also, the elastic pendulum is often studied [20, 21] by deriving formulas based on Taylor approximation. Such results have a narrow (and often unspecified) range of validity. Also, it is difficult to increase their precision. We suggest a complementary approach. After having derived the condition of stability of vertical oscillations in a closed form we use the numerical continuation technique to find the border of the stable region in the parameter space. Each step in this continuation involves numerical solution of the variational equations. Both the numerical integration and the numerical continuation can be done with any desired precision and the accuracy of the results can be checked in each step. We have used 30 decimal digits for numerical integration and 16 decimal digits for continuation.

The computed border between the stable and the unstable region can be approximated by a straight line (after appropriate pre-processing) giving a useful approximate closed form stability condition: the vertical oscillation with period a is stable if

$$|a| < |a_C(p)|$$

where

$$a_C(p) \doteq (3p)^{\frac{2}{3}} - 1.$$

ACKNOWLEDGMENTS

This work has been supported by the project MSM 6046137306 of the Czech Ministry of Education.

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