

$$T = 2\pi$$

$$\sum_m = \sum_{m \in \mathbb{Z}}$$

$$f(t) = \sum_m \hat{f}_m e^{imt}$$

$$\hat{f}_m = \frac{1}{T} \int_0^T f(t) e^{-imt} dt$$

Pro konvoluci platí

$$(f * g)(x) = \frac{1}{T} \int_0^T f(y) g(x - y) dy$$

a pro konvoluci Fourierových koeficientů

$$(\hat{f} * \hat{g})_m = \sum_k \hat{f}_k \hat{g}_{m-k}.$$

a)

Chceme ukázat, že

$$\widehat{f * g} = \hat{f} \hat{g},$$

tj.

$$(\widehat{f * g})_m = \hat{f}_m \hat{g}_m.$$

Označme levou stranu

$$\begin{aligned} L &= (\widehat{f * g})_m \\ &= \frac{1}{T} \int_0^T (f * g)(t) e^{-imt} dt \\ &= \frac{1}{T} \int_0^T \left[\frac{1}{T} \int_0^T f(y) g(t - y) dy \right] e^{-imt} dt \\ &= \frac{1}{T} \int_0^T f(y) \left[\frac{1}{T} \int_0^T g(t - y) e^{-imt} dt \right] dy. \end{aligned}$$

Provedeme substituci

$$\tau = t - y, \quad t = y + \tau.$$

Potom

$$\begin{aligned}
L &= \frac{1}{T} \int_0^T f(y) \left[\frac{1}{T} \int_0^T g(\tau) e^{-im(y+\tau)} d\tau \right] dy \\
&= \frac{1}{T} \int_0^T f(y) e^{-imy} dy \frac{1}{T} \int_0^T g(\tau) e^{-im\tau} d\tau \\
&= \widehat{f}_m \widehat{g}_m = R.
\end{aligned}$$

Tedy

$$\boxed{(\widehat{f * g})_m = \widehat{f}_m \widehat{g}_m}.$$

b)

Nyní chceme ukázat, že

$$\widehat{fg} = \widehat{f} * \widehat{g},$$

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$$(\widehat{fg})_m = (\widehat{f} * \widehat{g})_m.$$

Pro $k, n, m \in \mathbb{Z}$ platí ortogonalita exponenciál

$$\frac{1}{T} \int_0^T e^{it(k+n-m)} dt = \begin{cases} 1, & k+n=m, \\ 0, & k+n \neq m. \end{cases}$$

Proto

$$\begin{aligned}
L &= (\widehat{fg})_m \\
&= \frac{1}{T} \int_0^T f(t) g(t) e^{-imt} dt \\
&= \frac{1}{T} \int_0^T \left(\sum_k \widehat{f}_k e^{ikt} \right) \left(\sum_n \widehat{g}_n e^{int} \right) e^{-imt} dt \\
&= \sum_k \sum_n \widehat{f}_k \widehat{g}_n \frac{1}{T} \int_0^T e^{it(k+n-m)} dt \\
&= \sum_k \widehat{f}_k \widehat{g}_{m-k} \\
&= (\widehat{f} * \widehat{g})_m.
\end{aligned}$$

Tedy

$$\boxed{(\widehat{fg})_m = (\widehat{f} * \widehat{g})_m}.$$