Ukážeme si 3 způsoby, jak spočítat integrál

$$I_0 = \int \frac{1}{(1+x^2)^2} dx$$

1. Pomocí substituce

$$I_0 = \int \frac{1}{(1+x^2)^2} dx = \int \frac{1}{(1+x^2)} \frac{dx}{(1+x^2)} = \left| \begin{array}{ccc} x & = & \tan t \\ t & = & \arctan x \\ dt & = & \frac{dx}{1+x^2} \end{array} \right| = \int \frac{1}{1+\tan^2 t} dt = \int \frac{1}{1+\frac{\sin^2 t}{\cos^2 t}} dt =$$

$$= \int \cos^2 t \, dt = \frac{1}{2} \int (1+\cos 2t) dt = \frac{1}{2}(t + \frac{1}{2}\sin 2t) = \frac{1}{2}(t + \sin t \cos t) = \frac{1}{2}(t + \frac{\tan t}{1+\tan^2 t}) = \frac{1}{2}(\arctan x + \frac{x}{1+x^2}).$$

2. Rozkladem v komplexním oboru (použijeme imaginární jednotku i, pro kterou platí $i^2 = -1$)

$$\frac{1}{1+x^2} = \frac{i}{2}(\frac{1}{x+i} - \frac{1}{x-i})$$

$$(\frac{1}{1+x^2})^2 = (\frac{i}{2}(\frac{1}{x+i} - \frac{1}{x-i}))^2 = -\frac{1}{4}(\frac{1}{(x+i)^2} + \frac{1}{(x-i)^2} - \frac{2}{x^2+1})$$

$$I_0 = \int \frac{1}{(1+x^2)^2} dx = \int -\frac{1}{4}(\frac{1}{(x+i)^2} + \frac{1}{(x-i)^2} - \frac{2}{x^2+1}) dx = \frac{1}{4}(\frac{1}{x+i} + \frac{1}{x-i} + 2\arctan x = \frac{1}{2}(\arctan x + \frac{x}{1+x^2}).$$

3. Pomocí vneseného parametru:
Víme, že

$$\int \frac{1}{1+x^2} dx = \arctan x$$

a pro $a > 0$ použitím substituce dostaneme

$$\int \frac{1}{a+x^2} dx = \left| \begin{array}{ccc} x & = & \sqrt{a}\ t \\ dx & = & \sqrt{a}\ dt \end{array} \right| = a^{-\frac{1}{2}} \int \frac{1}{1+t^2} dt = a^{-\frac{1}{2}} \arctan t = a^{-\frac{1}{2}} \arctan(a^{-\frac{1}{2}} x).$$

Nyní zderivujeme vztah

$$\int \frac{1}{a+x^2} dx = a^{-\frac{1}{2}} \arctan(a^{-\frac{1}{2}} x)$$

podle a a dostaneme

$$-\int \frac{1}{(a+x^2)^2} dx = -\frac{1}{2} a^{-\frac{3}{2}} \arctan(a^{-\frac{1}{2}} x) + a^{-\frac{1}{2}} \frac{-\frac{1}{2} x a^{-\frac{3}{2}}}{1+a^{-1}x^2}$$

a po dosazení $a = 1$ dostaneme (po vynásobení -1) opět

$$I_0 = \int \frac{1}{(1+x^2)^2} dx = \frac{1}{2}(\arctan x + \frac{x}{1+x^2}).$$

A nyní již snadno spočítáme následující tři integrály:

$$I_1 = \int \frac{x}{(1+x^2)^2} dx = \left| \begin{array}{ccc} t & = & 1+x^2 \\ dt & = & 2x dx \end{array} \right| = \frac{1}{2} \int \frac{1}{t^2} dt = -\frac{1}{2}\frac{1}{t} = -\frac{1}{2}\frac{1}{1+x^2}.$$

$$I_2 = \int \frac{x^2}{(1+x^2)^2} dx = \int \frac{x^2+1-1}{(1+x^2)^2} dx = \int (\frac{1}{1+x^2} - \frac{1}{(1+x^2)^2}) dx = \arctan x - \frac{1}{2}(\arctan x + \frac{x}{1+x^2}) = \frac{1}{2}(\arctan x - \frac{x}{1+x^2}).$$

$$I_3 = \int \frac{x^3}{(1+x^2)^2} dx = \left| \begin{array}{ccc} t & = & 1+x^2 \\ dt & = & 2x dx \end{array} \right| = \frac{1}{2} \int \frac{t-1}{t^2} dt = \frac{1}{2}(\ln t + \frac{1}{t}) = \frac{1}{2}(\ln(1+x^2) + \frac{1}{1+x^2}).$$

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