An Interesting Application of the Intermediate Value Theorem: A Simple Proof of Sharkovsky's Theorem

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Throughout this note, I is a compact interval, and $f : I \to I$ is a continuous map. For each integer $n \geq 1$, let f^n be defined by: $f^1 = f$ and $f^n = f \circ f^{n-1}$ when $n \geq 2$. For x_0 in I, we call the set $\{x_0, f(x_0), f^2(x_0), \cdots\}$ the orbit of x_0 with respect to f and call x_0 a periodic point of f with least period m or a period-m point of f if $f^m(x_0) = x_0$ and $f^i(x_0) \neq x_0$ when $0 < i < m$. If $f(x_0) = x_0$, then we call x_0 a fixed point of f.

The celebrated Sharkovsky's cycle coexistence theorem [**31**] can be stated as follows:

Theorem (Sharkovsky[27, 28, 31]) *Let the Sharkovsky's ordering of the natural numbers be defined as follows:*

$$3 \prec 5 \prec 7 \prec 9 \prec \cdots \prec 2 \cdot 3 \prec 2 \cdot 5 \prec 2 \cdot 7 \prec 2 \cdot 9 \prec \cdots \prec 2^2 \cdot 3 \prec 2^2 \cdot 5 \prec 2^2 \cdot 7 \prec 2^2 \cdot 9 \prec \cdots$$
$$\cdots \prec 2^3 \prec 2^2 \prec 2 \prec 1.$$

Then the following three statements hold:

(1) *If f has a period-m point and if $m \prec n$, then f also has a period-n point.*

(2) *For each positive integer n there exists a continuous map from I into itself that has a period-n point but has no period-m point for any m with $m \prec n$.*

(3) *There exists a continuous map from I into itself that has a period-2^i point for $i = 0, 1, 2, \ldots$ but has no periodic point of any other period.*

This note is mainly excerpted from [**17**]. To make it self-contained, we include the following two well-known results.

Lemma 1. If $f^n(x_0) = x_0$, then the least period of x_0 with respect to f divides n.

Proof. Let m denote the least period of x_0 with respect to f and write $n = km + r$ with $0 \leq r < m$. Then $x_0 = f^n(x_0) = f^{km+r}(x_0) = f^r(f^{km}(x_0)) = f^r(x_0)$. Since m is the smallest positive integer such that $f^m(x_0) = x_0$, we must have $r = 0$. Therefore, m divides n. ∎

Lemma 2. *Let $k, m, n,$ and s be positive integers. Then the following statements hold:*

(i) *If x_0 is a periodic point of f with least period m, then it is a periodic point of f^n with least period $m/(m,n)$, where (m,n) is the greatest common divisor of m and n.*

(ii) *If x_0 is a periodic point of f^n with least period k, then it is a periodic point of f with least period kn/s, where s divides n and is relatively prime to k. In particular, if $f^{2^{k-1}}$ has a period-$(2\cdot m)$ point for some $k \geq 2$ and $m \geq 1$, then f has a period-$(2^k \cdot m)$ point.*

Proof. (i) Let x_0 be a period-t point of f^n. Then m divides nt since $x_0 = (f^n)^t(x_0) = f^{nt}(x_0)$. So, $\frac{m}{(m,n)}$ divides $\frac{n}{(m,n)} \cdot t$. Since $\frac{m}{(m,n)}$ and $\frac{n}{(m,n)}$ are coprime, $\frac{m}{(m,n)}$ divides t. Furthermore, $(f^n)^{(m/(m,n))}(x_0) = (f^m)^{(n/(m,n))}(x_0) = x_0$. Thus, t divides $\frac{m}{(m,n)}$. This shows that $t = \frac{m}{(m,n)}$.

(ii) Since $x_0 = (f^n)^k(x_0) = f^{kn}(x_0)$, the least peirod of x_0 under f is $\frac{kn}{s}$ for some positive integer s. By (i), $(\frac{kn}{s})/((\frac{kn}{s}), n) = k$. So, $\frac{n}{s} = ((\frac{n}{s})k, n)$ (which is an integer) $= ((\frac{n}{s})k, (\frac{n}{s})s) = (\frac{n}{s})(k,s)$. This shows that s divides n and $(s,k) = 1$. ■

Following **[35]**, we first prove the following three statements:

(a) if f has a period-m point with $m \geq 2$, then f has a period-2 point and a fixed point;

(b) if f has a period-m point with $m \geq 3$ and odd, then f has a period-$(m+2)$ point; and

(c) if f has a period-m point with $m \geq 3$ and odd, then f has periodic points of all even periods.

Let P be a period-m orbit of f with $m \geq 2$ and let $b = f^{m-1}(\min P)$. Then $f(b) = \min P < b$. If $f(x) < b$ on $[\min P, b]$, then, $(\min P \leq)$ $f^i(\min P) < b$ for all $i \geq 1$, contradicting the fact that $f^{m-1}(\min P) = b$. So, there is a point a in $[\min P, b]$ such that $f(a) \geq b$. Thus, f has a fixed point z in $[a,b]$. Now suppose $m \geq 3$ and let v be a point in $[a,z]$ such that $f(v) = b$. Since $f^2(\min P) > \min P$ and $f^2(v) = \min P < v$, the point $y = \max\{\min P \leq x \leq v : f^2(x) = x\}$ exists. Furthermore, $f(x) > z$ on $[y,v]$ and $f^2(x) < x$ on $(y,v]$. Therefore, y is a period-2 point of f. (a) is proved.

For the proofs of (b) and (c), we assume that $m \geq 3$ is odd and note that $f(x) > z > x > f^2(x)$ on $(y,v]$. Since $f^{m+2}(y) = f(y) > y$ and $f^{m+2}(v) = f^m(\min P) = \min P < v$, the point $p_{m+2} = \min\{y \leq x \leq v : f^{m+2}(x) = x\}$ exists. Let k denote the least period of p_{m+2} with respect to f. Then $k > 1$ and, by Lemma 1, k divides $m+2$. So, k is odd. If $k < m+2$, then since $f^{k+2}(y) = f(y) > y$ and $f^{k+2}(p_{m+2}) = (f^2)(f^k(p_{m+2})) = f^2(p_{m+2}) < p_{m+2}$, there is a point w_{k+2} in (y, p_{m+2}) such that $f^{k+2}(w_{k+2}) = w_{k+2}$. Inductively, there exist points

$$y < \cdots < w_{m+2} < w_m < w_{m-2} < \cdots < w_{k+4} < w_{k+2} < p_{m+2} < v$$

such that $f^{k+2i}(w_{k+2i}) = w_{k+2i}$ for all $i \geq 1$. In particular, $f^{m+2}(w_{m+2}) = w_{m+2}$ and $y < w_{m+2} < p_{m+2}$, contradicting the fact that p_{m+2} is the *smallest* point in (y,v) which satisfies $f^{m+2}(x) = x$. Therefore, $k = m+2$. This establishes (b).

We now prove (c). Let

$$z_0 = \min\{v \leq x \leq z : f^2(x) = x\}.$$

Then $f^2(x) < x$ and $f(x) > z$ on (v, z_0) and so also on (y, z_0). If $f^2(x) < z_0$ whenever $\min P \leq x < z_0$, then we have $\min P \leq f^{2i}(\min P) < z_0$ for all $i \geq 1$ which contradicts the fact that $(f^2)^{(m-1)/2}(\min P) = b > z_0$. Since $f^2(x) < x < z_0$ on (y, z_0), the point

$$d = \max\{\min P \le x \le y : f^2(x) = z_0\}$$

exists and $f(x) > z \ge z_0 > f^2(x)$ on (d, y). Therefore, $f(x) > z \ge z_0 > f^2(x)$ on (d, z_0). Let

$$u_1 = \min\{d \le x \le v : f^2(x) = d\}.$$

Then $d < f^2(x) < z_0$ on (d, u_1). Let c_1 be any point in (d, u_1) such that $f^2(c_1) = c_1$. Let

$$u_2 = \min\{d \le x \le c_1 : (f^2)^2(x) = d\}.$$

Then $d < (f^2)^2(x) < z_0$ on (d, u_2). Let c_2 be any point in (d, u_2) such that $(f^2)^2(c_2) = c_2$. Let

$$u_3 = \min\{d \le x \le c_2 : (f^2)^3(x) = d\}.$$

Then $d < (f^2)^3(x) < z_0$ on (d, u_3). Let c_3 be any point in (d, u_3) such that $(f^2)^3(c_3) = c_3$. Proceeding in this manner indefinitely, we obtain points

$$d < \cdots < c_n < u_n < \cdots < c_2 < u_2 < c_1 < u_1 < z_0$$

such that $d < (f^2)^n(x) < z_0$ on (d, u_n) and $(f^2)^n(c_n) = c_n$. Since $f(x) > z \ge z_0$ on (d, z_0), we have

$$f^i(c_n) < z_0 < f^j(c_n) \ \text{ for all even } \ i \ \text{ and all odd } \ j \ \text{ in } \ [0, 2n].$$

Therefore, each c_n is a period-$(2n)$ point of f. This proves (c).

We now prove (1), (2) and (3) of Sharkovsky's theorem.

If f has a period-m point with $m \ge 3$ and odd, then it follows from (b) that f has a period-$(m+2)$ point and, from (c) that f has periodic points of all even periods.

If f has a period-$(2 \cdot m)$ point with $m \ge 3$ and odd, then, by Lemma 2(i), f^2 has a period-m point. It follows from the above (or by (b) and (c)) that f^2 has a period-$(m+2)$ point and a period-$(2 \cdot 3)$ point. If f^2 has a period-$(m+2)$ point, then, by Lemma 2(ii),

$$f \ \text{ has either a period-}(m+2) \ \text{ point or a period-}(2 \cdot (m+2)) \ \text{ point.}$$

If f has a period-$(m+2)$ point, then it follows from (c) that f has a period-$(2 \cdot (m+2))$ point. In either case, f has a period-$(2 \cdot (m+2))$ point. On the other hand, if f^2 has a period-$(2 \cdot 3)$ point, then, by Lemma 2(ii), f has a period-$(2^2 \cdot 3)$ point. This shows that if f has a period-$(2 \cdot m)$ point with $m \ge 3$ and odd, then f has a period-$(2 \cdot (m+2))$ point and a period-$(2^2 \cdot 3)$ point.

Now if f has a period-$(2^k \cdot m)$ point with $m \ge 3$ and odd and if $k \ge 2$, then, by Lemma 2(i), $f^{2^{k-1}}$ has a period-$(2 \cdot m)$ point. It follows from the previous paragraph that $f^{2^{k-1}}$ has a period-$(2 \cdot (m+2))$ point and a period-$(2^2 \cdot 3)$ point. So, by Lemma 2(ii), f has a period-$(2^k \cdot (m+2))$ point and a period-$(2^{k+1} \cdot 3)$ point.

Furthermore, if f has a period-$(2^i \cdot m)$ point with $m \ge 3$ and odd and if $i \ge 0$, then, by Lemma 2(i), f^{2^i} has a period-m point. For each $\ell \ge i$, by Lemma 2(i), $f^{2^\ell} = (f^{2^i})^{2^{\ell-i}}$ has a period-m point and so, by (c), f^{2^ℓ} has a period-6 point. Thus, by Lemma 2(i), $f^{2^{\ell+1}}$ has a period-3 point and hence, by (a), has a period-2 point. This implies, by Lemma 2(ii), that f has a period-$2^{\ell+2}$ point for each $\ell \ge i$.

Finally, if f has a period-2^k point for some $k \geq 2$, then, by Lemma 2(i), $f^{2^{k-2}}$ has a period-4 point. By (a), $f^{2^{k-2}}$ has a period-2 point. By Lemma 2(ii), f has a period-2^{k-1} point and hence, by induction, f has a period-2^j point for each $j = 1, 2, \cdots, k-2$. Furthermore, it follows from (a) that f has a fixed point. This completes the proof of (1).

As for the existence proofs of (2) and (3) (we refer to **[17]** for some constructive examples), we let $g(x) : [0,1] \to [0,1]$ denote any continuous map that has at least one period-3 orbit and *finitely many* (≥ 1 by (1)) period-k orbits for each $k \geq 2$. For example, we can take $g(x)$ to be the tent map $g(x) = 1 - |2x - 1|$ (cf. **[11]**). We also let the truncated map $\hat{g}_{a,b}(x)$, where $0 \leq a < b \leq 1$, be defined on $[0,1]$ by

$$\hat{g}_{a,b}(x) = \begin{cases} b, & \text{if } g(x) > b; \\ g(x), & \text{if } a \leq g(x) \leq b; \\ a, & \text{if } g(x) < a. \end{cases}$$

The relationship between the maps $g(x)$ and $\hat{g}_{a,b}(x)$ is that the periodic orbits of $\hat{g}_{a,b}(x)$ are also periodic orbits of $g(x)$ with the same periods and, conversely, the periodic orbits of $g(x)$ which lie entirely in the interval $[a,b]$ are also periodic orbits of $\hat{g}_{a,b}(x)$ with the same periods. Consequently, if Q_k is a period-k orbit of $g(x)$, then it is also a period-k orbit of $\hat{g}_{\min Q_k, \max Q_k}(x)$. By (1), $\hat{g}_{\min Q_k, \max Q_k}(x)$ has a period-ℓ orbit for each ℓ with $k \prec \ell$. In other words, the interval $[\min Q_k, \max Q_k]$ contains a period-ℓ orbit of $g(x)$ for each ℓ with $k \prec \ell$. By assumption, for each integer $k \geq 2$, $g(x)$ has *finitely many* (≥ 1) period-k orbits. Among these *finitely many* period-k orbits, let

$$P_k \text{ be one with the } \textit{smallest} \text{ diameter } \max P_k - \min P_k.$$

For each x in $[0,1]$, let $\hat{g}_k(x) = \hat{g}_{a_k,b_k}(x)$, where $a_k = \min P_k$ and $b_k = \max P_k(x)$. Then it is easy to see that, for each $k \geq 2$, $\hat{g}_k(x)$ has exactly one period-k orbit (i.e., P_k) but has no period-j orbit for any j with $j \prec k$ in the Sharkovsky ordering. This, together with the constant maps, confirms (2).

By assumption, $g(x)$ has *finitely many* (≥ 1) period-2 orbits. Let δ denote the *smallest* diameter among these period-2 orbits. For every periodic orbit P of $g(x)$ with least period ≥ 3, it follows from (a) that $\hat{g}_{\min P, \max P}(x)$ has a period-2 orbit. So, $\max P - \min P \geq \delta > 0$. Now let Q_3 be any period-3 orbit of $g(x)$ of smallest diameter. Then $[\min Q_3, \max Q_3]$ contains finitely many period-6 orbits of $g(x)$ among which one, say Q_6, is of smallest diameter. Sinilarly, $[\min Q_6, \max Q_6]$ contains finitely many period-12 orbits of $g(x)$ among which one, say Q_{12}, is of smallest diameter. We continue the process inductively. Let

$$q_0 = \sup\{\min Q_{2^n \cdot 3} : n \geq 0\} \text{ and } q_1 = \inf\{\max Q_{2^n \cdot 3} : n \geq 0\}$$

and let $\hat{g}_\infty(x) = \hat{g}_{q_0,q_1}(x)$ for all $0 \leq x \leq 1$. If $\hat{g}_\infty(x)$ had a period-$(2^i \cdot m)$ orbit for some $i \geq 0$ and some odd $m \geq 3$, then, by (1), $\hat{g}_\infty(x)$ has a period-$(2^{i+1} \cdot 3)$ orbit, say $\hat{Q}_{2^{i+1} \cdot 3}$. Since $\hat{Q}_{2^{i+1} \cdot 3} \subset [q_0, q_1] \subsetneq [\min Q_{2^{i+1} \cdot 3}, \max Q_{2^{i+1} \cdot 3}]$, $\hat{Q}_{2^{i+1} \cdot 3}$ is also a period-$(2^{i+1} \cdot 3)$ orbit of $g(x)$ with diameter *strictly smaller* than that of $Q_{2^{i+1} \cdot 3}$. This is a contradiction. So, $\hat{g}_\infty(x)$ has no periodic orbit of period not a power of 2. On the other hand, for each $k \geq 0$, the map $g(x)$ has finitely many period-2^k orbits. If each such orbit had an *exceptional* point which is not in the interval $[q_0, q_1]$, then it is clear that we can find an $n \geq 1$ such that the interval $[\min Q_{2^n \cdot 3}, \max Q_{2^n \cdot 3}]$ contains none of these *exceptional* points which implies that $[\min Q_{2^n \cdot 3}, \max Q_{2^n \cdot 3}]$ contains no period-2^k *orbits* of $g(x)$. Consequently, the map $\hat{g}_{s_n,t_n}(x)$, where $s_n = \min Q_{2^n \cdot 3}, t_n = \max Q_{2^n \cdot 3}$, has no period-$2^k$ *orbits* and yet it has a period-$(2^n \cdot 3)$ orbit, i.e., $Q_{2^n \cdot 3}$. This contradicts (1). Therefore,the map $\hat{g}_\infty(x)$ is an example for (3).

References

[1] L. Alsedà, On the number of periodic orbits of continuous mappings on the interval, *Pub. Mat. UAB* **25** (1981), 97-106.

[2] L. Alsedà, J. Llibre and M. Misiurewicz, *Combinatorial dynamics and entropy in dimension one*, Second edition, Advanced Series in Nonlinear Dynamics, **5**, World Scientific, River Edge, NJ, 2000.

[3] A. Arneodo, P. Ferrero and C. Tresser, Sharkovskiĭ's order for the appearance of superstable cycles of one-parameter families of simple real maps: an elementary proof, *Comm. Pure Appl. Math.* **37** (1984), 13-17.

[4] R. Barton and K. Burns, A simple special case of Sharkovskii's theorem, *Amer. Math. Monthly* **107** (2000), 932-933.

[5] C. Bernhardt, A proof of Sharkovsky's theorem, *J. Diff. Equ. Appl.* **9** (2003), 373-379.

[6] N. Bhatia and W. Egerland, New proof and extension of Sarkovskii's theorem, *Far East J. Math. Sci.* Special volume, Part I, 1996, 53-67.

[7] L. Block, Stability of periodic orbits in the theorem of Šarkovskii, *Proc. Amer. Math. Soc.* **81** (1981), 333-336.

[8] L. Block and W. A. Coppel, *Dynamics in one dimension*, Lecture Notes in Mathematics, **1513**, Springer-Verlag, Berlin, 1992.

[9] L. Block, J. Guckenheimer, M. Misiurewicz and L.-S. Young, Periodic points and topological entropy of one-dimensional maps, in *Global Theory of Dynamical Systems*, Lecture Notes in Math., **819**, Springer-Verlag, Berlin, 1980, pp. 18-34.

[10] U. Burkart, Interval mapping graphs and periodic points of continuous functions, *J. Combin. Theory Ser. B* **32** (1982), 57-68.

[11] K. Burns and B. Hasselblatt, The Sharkovsky theorem: A natural direct proof, *Amer. Math. Monthly* **118** (2011), 229-244.

[12] K. Ciesielski and Z. Pogoda, On ordering the natural numbers or the Sharkovski theorem, *Amer. Math. Monthly* **115** (2008), 159-165.

[13] W. A. Coppel, The solution of equations by iteration, *Proc. Cambridge Philos. Soc.* bf 51 (1955), 41-43.

[14] B.-S. Du, The minimal number of periodic orbits of periods guaranteed in Sharkovskii's theorem, *Bull. Austral. Math. Soc.* **31** (1985), 89-103. Corrigendum: **32** (1985), 159.

[15] B.-S. Du, A simple proof of Sharkovsky's theorem, *Amer. Math. Monthly* **111** (2004), 595-599.

[16] B.-S. Du, A simple proof of Sharkovsky's theorem revisited, *Amer. Math. Monthly* **114** (2007), 152-155.

[17] B.-S. Du, A simple proof of Sharkovsky's theorem rerevisited, arXiv:0711.3892v9.

[18] S. Elaydi, On a converse of Sharkovsky's theorem, *Amer. Math. Monthly* **103** (1996), 386-392.

[19] B. Gaweł, On the theorems of Šarkovskiĭ and Štefan on cycles, *Proc. Amer. Math. Soc.* **107** (1989), 125-132.

[20] J. Guckenheimer, On the bifurcation of maps of the interval, *Invent. Math.* **39** (1977), 165-178.

[21] C.-W. Ho and C. Morris, A graph-theoretic proof of Sharkovsky's theorem on the periodic points of continuous functions, *Pacific J. Math.* **96** (1981), 361-370.

[22] X. C. Huang, From intermediate value theorem to chaos, *Math. Mag.* **65** (1992), 91-103.

[23] H. Kaplan, A cartoon-assisted proof of Sarkowskii's theorem, *Amer. J. Phys.* **55** (1987), 1023-1032.

[24] J. P. Keener, The Sarkovskiĭ sequence and stable periodic orbits of maps of the interval, *SIAM J. Numer. Anal.* **23** (1986), 976-985.

[25] T.-Y. Li and J. A. Yorke, Period three implies chaos, *Amer. Math. Monthly* **82**(1975), 985-992.

[26] J.-H. Mai, 3-periodic orbit implying 6831726876986508 85-periodic orbits-infimums of numbers of periodic orbits in continuous functions, *Sci. China (Ser. A)* **34** (1991), 1194-1204.

[27] M. Misiurewicz, Remarks on Sharkovsky's theorem, *Amer. Math. Monthly* **104** (1997), 846-847.

[28] M. Misiurewicz, http://www.scholarpedia.org/article/Combinatorial dynamics

[29] M. Osikawa and Y. Oono, Chaos in C^0-endomorphism of interval, *Publ. RIMS Kyoto Univ.* 17 (1981), 165-177.

[30] S. Patinkin, Stirring our way to Sharkovsky's theorem, *Bull. Austral. Math. Soc.* **56** (1997), 453-458.

[31] A. N. Sharkovsky, Coexistence of cycles of a continuous map of a line into itself, *Ukrain. Mat. Zh.* **16** (1964) 61-71 (Russian); English translation, *Internat. J. Bifur. Chaos Appl. Sci. Engrg.* **5** (1995), 1263-1273.

[32] A. N. Sharkovsky, S. F. Kolyada, A. G. Sivak and V. V. Fedorenko, *Dynamics of one-dimensional maps*, Math. and its Applications, **407**, Kluwer Academic, Dordrecht, 1997.

[33] A. N. Sharkovsky, http://www.scholarpedia.org/article/Sharkovsky ordering

[34] P. Štefan, A theorem of Šarkovskii on the existence of periodic orbits of continuous endomorphisms of the real line, *Comm. Math. Phys.* **54** (1977), 237-248.

[35] P. D. Straffin Jr., Periodic points of continuous functions, *Math. Mag.* **51** (1978), 99-105.

[36] J.-C. Xiong, A simple proof of Sharkovskiĭ's theorem on the existence of periodic points of continuous self-maps of an interval, *J. China Univ. Sci. Tech.* **12** (1982), 17-20 (Chinese).

[37] J. Z. Zhang and L. Yang, Some theorems on the Sarkovskiĭ order, *Adv. in Math.* (Beijing) **16** (1987), 33-48 (Chinese).